Representations in characteristic $p>0$

I. Summary of results so far

$G \supset B \supset T$ $\quad X(T) \supset X^+$ dominant

G conn. reduc. gp, B Borel, T max torus

$\forall \lambda \in X^+$, $\Bbbk_\lambda$ = 1-dim'l B-rep on which T acts by λ

$V_\lambda = \mathrm{ind}_B^G \Bbbk_\lambda$

L_λ = unique irred. subrep of V_λ

Thm. (Classif. of irred. reps) There is a bijection

$$\left\{\begin{array}{l}\text{isom. classes of} \\ \text{irred reps}\end{array}\right\} \longleftrightarrow X^+$$

$$L_\lambda \longleftrightarrow \lambda$$

Thm For $\Bbbk = \mathbb{C}$

① $L_\lambda = V_\lambda$ (so $\mathrm{ch}\, L_\lambda$ = Weyl character formula)

② Every rep of G is completely reducible, i.e. a $\oplus$ of irred reps.

Both parts are false when $\mathrm{char}\, \Bbbk = p > 0$

Example SL_2 / $\Bbbk$ of char. $p>0$.

Recall: $\forall n \geq 0$

$V_n = \mathrm{span}\{x^n, x^{n-1}y, \dots, y^n\}$

action: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot f(x,y) = f(ax+cy, bx+dy)$

L_n = smallest subrep containing x^n.

Look at the case $n=p$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot x^p = (ax+cy)^p = a^p x^p + c^p y^p \quad \text{because char } \Bbbk = p$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot y^p = b^p x^p + d^p y^p.$$

So: $\mathrm{span}\{x^p, y^p\} \subset V_p$ is a subrep.

This is L_p. $\quad L_p \neq V_p$.

Look at the short exact sequence

$$0 \to L_p \hookrightarrow V_p \twoheadrightarrow V_p/L_p \to 0$$

Exercise This short exact seq. does not split. So: $V_p \not\cong L_p \oplus V_p/L_p$.

V_p is NOT completely reducible.

$\dim L_p = 2 \qquad \dim V_p = p+1$.

Similar: $L_1, L_p, L_{p^2}, L_{p^3}, \dots$ all 2-dim'l.

$L_1 = V_1$

Explain this.

II. Frobenius twist

Observation: There is a group homom
$$Fr: GL_n \longrightarrow GL_n$$
given by
$$Fr\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix}\right) = \begin{bmatrix} a_{11}^p & a_{12}^p & \cdots & a_{1n}^p \\ \vdots & & & \\ a_{n1}^p & & \cdots & a_{nn}^p \end{bmatrix}$$

(This is NOT the same as raising the whole matrix to the p^{th} power)

Note Fr is a bijection
(In an alg closed field of char p, every element has a unique p^{th} root).
but NOT an isom of alg gps.
(Fr^{-1} is NOT a morphism of varieties).

Defn. Let V be a G-rep, $\pi: G \to GL(V)$. The (first) Frobenius twist of V is the rep. $V^{(1)}$ given by:
- vector space is still V
- action is $\pi^{(1)} = Fr \circ \pi: G \to GL(V)$.

Can repeat this construction: $V^{(2)}, V^{(3)}, \ldots$

Examples

1) $\chi: \mathbb{G}_m \to \mathbb{G}_m$, $\chi(z) = z^n$
   $\mathbb{G}_m = GL_1$

   $\chi^{(1)}: \mathbb{G}_m \to \mathbb{G}_m$ $\chi(z) = z^{pn}$

2) T a torus, $\lambda \in X(T)$ $\lambda: T \to \mathbb{G}_m$
   $\lambda^{(1)} = p\lambda \in X(T)$.

3) V any T-rep.
   Suppose $\mathrm{ch}\, V = \sum_{\lambda \in X(T)} a_\lambda e^\lambda$.

   Then $\mathrm{ch}\, V^{(1)} = \sum_{\lambda \in X(T)} a_\lambda e^{p\lambda}$

Lemma. If V is an irred G-rep, then $V^{(1)}$ is also irred.

Proof: Exercise.

Prop. G conn reductive gp. For $\lambda \in X^+$, $L_\lambda^{(1)} \simeq L_{p\lambda}$.

Proof. L_λ is the unique irred rep s.t. $\mathrm{ch}\, L_\lambda = e^\lambda + \underbrace{(\text{lower terms})}_{\text{with respt to } \preceq}$

$L_\lambda^{(1)}$ is irred, has character $e^{p\lambda} + (\text{lower terms})$. So $L_\lambda^{(1)} \simeq L_{p\lambda}$. $\square$

III. Steinberg tensor product theorem

$G \supset B \supset T \qquad X(T) \supset X^+$

Also: can define coroots $\subset Y(T)$
$\cup$
simple coroots $\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_r^\vee$

Pairing $\langle , \rangle : X(T) \times Y(T) \longrightarrow \mathbb{Z}$.

Fact. For $\lambda \in X(T)$,
λ is dominant $\Leftrightarrow$ $\langle \lambda, \alpha_i^\vee \rangle \geq 0$
for $i=1, \dots, r$.

Defn. $\lambda \in X(T)$ is said to be p-restricted if $0 \leq \langle \lambda, \alpha_i^\vee \rangle < p$ for $i=1,\dots,r$.

Notation: X_1^+ = set of p-restricted weights $\subset X^+$.

Fact $\forall \lambda \in X^+$, $\exists$ unique expr

$$(*) \quad \lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \cdots + p^k\lambda_k$$

where $\lambda_0, \lambda_1, \dots, \lambda_k \in X_1^+$.

Thm (Steinberg) version 1: If $\lambda \in X_1^+$, $\mu \in X^+$ then $L_{\lambda + p\mu} \simeq L_\lambda \otimes L_\mu^{(1)}$

Version 2: For $(*)$

$$L_\lambda \simeq L_{\lambda_0} \otimes L_{\lambda_1}^{(1)} \otimes L_{\lambda_2}^{(2)} \otimes \cdots \otimes L_{\lambda_k}^{(k)}$$

IV. Tilting modules

- Concept only introduced around 1990
- Very good homological properties.

Defn. G conn. reductive gp.
Let V be a G-rep. A good filtration of V is a sequence

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = V$$

of subreps such that each

$$M_i / M_{i-1} \simeq V_{\lambda_i} = \mathrm{ind}_B^G \Bbbk_{\lambda_i}$$

for some $\lambda_1, \dots, \lambda_k \in X^+$.

NOT all reps have good filtrations.

Defn. A G-rep ~~is~~ V is said to be tilting if both V and V^* have good filtrations

dual (or contragredient) rep to V.

$V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$

G acts on V^* by $(\underset{V^*}{g \cdot f})(v) = f(g^{-1} \cdot v)$

Example: $SL_2 / \Bbbk$ of char. p.

Fact: $\exists$ rep. V of dimension ~~2p+1~~ $2p$ with a filtration

$$
\begin{array}{ccccccc}
 & & & & V_p^* & & \\
 & & & & \shortparallel & & \\
0 = M_0 & \subset & M_1 & \subset & M_2 & \subset & M_3 = V \\
 & & \shortparallel & & & & \\
 & & L_{p-2} & & M_2/M_1 & & M_3/M_2 \\
 & & \wr\shortparallel & & \simeq L_p \simeq L_1^{(1)} & & \wr\shortparallel \\
 & & V_{p-2} & & & & L_{p-2} \\
 & & & & & & \wr\shortparallel \\
 & & & & & & V_{p-2}^*
\end{array}
$$

Also: $M_3/M_1 \simeq V_p$.

$0 \subset M_1 \subset M_3 = V$ is a good filtn.

$0 \subset M_2 \subset M_3 = V$ is a dual good filtn.

Prop. Given two reps V_1 & V_2, $V_1 \oplus V_2$ is tilting $\Leftrightarrow$ V_1 and V_2 are tilting

Defn. An indecomposable tilting module is a tilting module that cannot be written as a $\oplus$ of smaller tilting modules

Thm. (Classif. of indecomp. tilting) - Ringel, Donkin.
For each $\lambda \in X^+$, $\exists$ unique (up to isom) indecomp tilting T_λ s.t. $\mathrm{ch}\, T_\lambda = e^\lambda + (\text{lower weights})$.

So: bijection

$$\left\{ \begin{array}{c} \text{isom classes of} \\ \text{indecomp tilting} \end{array} \right\} \longleftrightarrow X^+.$$

Note: Every tilting rep of G is "tilting-completely reducible" i.e. $\oplus$ of indecomp tilting.

V. Character Formulas

Problem (very old):

- Compute $\mathrm{ch}\, L_\lambda$.

If $\Bbbk = \mathbb{C}$: Weyl character formula (1920's)

If char $\Bbbk = p > 0$, NOT even a conj until ~1980: "Lusztig conjecture"

- Proved in the 1990s for $p \gg 0$.
- Expected to hold for $p > \underbrace{n}_{\text{Coxeter number}}$ for GL_n.
- Williamson 2013: counterexamples in GL_n with $p > n$.

More recent problem:

- Compute $\mathrm{ch}\, T_\lambda$.
- related to $\mathrm{ch}\, L_\lambda$ by Andersen, Riche-Williamson, Sobaje
- Formula conj by R-W
  - Proved for $p >$ Coxeter number (n for GL_n) by Achar, Makisumi, $\underbrace{\text{Riche, Williamson}}$
- Proved for all p by March 2020