Measure equivalence superrigidity for some generalized Higman groups

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QI and ME classification

$f : X_1 \to X_2$ is a quasi-isometry iff there are constants $L, A > 0$ s.t.

1. $L^{-1}d(x,y) - A \leqslant d(f(x), f(y)) \leqslant Ld(x,y) + A$ for all $x, y \in X_1$.
2. $f(X_1)$ is A-dense in X_2.

(QI classification): classify finitely generated groups up to quasi-isometry.

(Gromov) Two f.g. G and H are quasi-isometric iff there exist commuting, properly discontinuous actions of G and H on some locally compact space X, such that the action of each of the groups G and H is cocompact.

X is the set of all (L, A) quasi-isometrics from G to H, equipped with the topology of pointwise convergence.

A topological coupling between G and H is an action of $G \times H$ on a locally compact space X by homeomorphism such that the action of each factor is properly discontinuous and cocompact.

Definition

A *measure equivalent coupling* between two countable groups G and H is a measurable and measure-preserving action of $G \times H$ on some measure space such that the action of each factor is free and admits finite measure fundamental domains.
Two countable groups G and H are measure equivalent (ME) if there is a ME coupling between them.

G is ME to its f.i. subgroup, as well as quotients of G by finite normal subgroups.

Two lattices in the same Lie group are ME. (lattices are discrete subgroups with finite covolume)

e.g. F_2 and $\pi_1(S_g)$ $(g \geqslant 2)$ are ME, as they are lattices in $\mathrm{Isom}(\mathbb{H}^2)$.

ME from the viewpoint of ergodic theory

In this slide we consider probability measure preserving (p.m.p.), ergodic action on probability measure space (X, μ).

Definition

Two p.m.p. actions $H \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ are orbit equivalence (OE) if there is a measure space isomorphism $T : (X, \mu) \to (Y, \nu)$ sending H-orbits to G-orbits.

Definition

Two countable groups are orbit equivalence (OE) if they admit free, ergodic, p.m.p. actions on probability spaces that are OE.

Definition

Two p.m.p. actions $H \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ are stably orbit equivalence (SOE) if there is positive measure subsets $X' \subset X$ and $Y' \subset Y$ and a measure scaling isomorphism $T' : X' \to Y'$ such that T' sends H-orbits in X' to G-orbit in Y'.

ME from the viewpoint of ergodic theory

Observation (Furman): Two countable groups H and G are ME iff they admit free, ergodic, p.m.p. actions $H \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ that are SOE.

More examples of ME:

Theorem (Ornstein-Weiss 1980)

Any two ergodic p.m.p. actions of any two infinite countable amenable groups are OE.

Corollary: any two countable infinite amenable groups are ME.

Reminder: A discrete group G is amenable if there exists a finitely additive, left invariant probability measure on G.
Examples: finite groups, abelian groups, closed under taking subgroups, forming quotients, and forming extensions.
E.g. $\mathbb{Z}$ is ME to $\mathbb{Z}^2$.

1. If H_i is OE to G_i for $1 \leqslant i \leqslant n$, then $H_1 \times \cdots \times H_n$ is OE to $G_1 \times \cdots \times G_n$.
2. (Gaboriau) If H_i is OE to G_i for $1 \leqslant i \leqslant n$, then $H_1 * \cdots * H_n$ is OE to $G_1 * \cdots * G_n$.

Take Γ be a finite simple graph. For each vertex $v \in \Gamma$, we associated a group G_v. The graph product of $\{G_v\}_{v \in V\Gamma}$ over Γ is defined to be

$$\ast G_v / \{[G_v, G_w] = 1 \text{ if } v, w \text{ are adjacent}\}$$

1. If Γ is a complete graph, the graph product is the direct sum.
2. If Γ is discrete, the graph product is a free product.
3. If each G_v is $\mathbb{Z}$, then the graph product is a right-angled Artin group.

Observation (Horbez-H.) Two graph products over the same graph are OE if their vertex groups are OE.

Cor: any right-angled Artin group is OE (hence ME) to a graph of products any infinite amenable groups with the same underlying graph.

Rigidity and invariants

Theorem (Furman 1999)

Let G' be a higher rank simple Lie group, and let $G \leqslant G'$ be an irreducible lattice. Then any countable group H measure equivalent to G is virtually a lattice in G'.

e.g. take $G' = SL(n, \mathbb{R})$ and $G = SL(n, \mathbb{Z})$.

Theorem (Kida 2010)

Outside a few sporadic cases, any countable group ME to a mapping class group G of a surface is virtually G.

Theorem (Guirardel-Horbez 2021)

Any countable group ME to $Out(F_n)$ $(n \geqslant 3)$ is virtually $Out(F_n)$.

ME invariants: amenability, property (T), Haagerup property

A particular strategy towards ME/QI rigidity

Let G be a mapping class group of a genus 2 closed surface. We want to understand any f.g. groups QI to G (or any countable group ME to G).

- (Step 0) Reduce to the study of self-quasi-isometry or self ME-coupling of G.
- (Step 1) Showing that each self-quasi-isometry or self ME-coupling "preserves" the collection of curve stabilizers in an appropriate sense, and preserves the intersection pattern of curve stabilizers. Thus self-quasi-isometry or self ME-coupling induces automorphisms of the curve graph.
- (Step 2) Use a rigidity theorem by Ivanov on curve graph, i.e. any automorphism of the curve graph is induced by a mapping class.

Summary: We need to find a robust collection of subgroups which are QI or ME invariants.

Preservation of subgroups

Given a group G and a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. What does it mean by a self ME-coupling "preserve" these subgroups?

Given two free, p.m.p., ergodic $G \curvearrowright X$ and $G \curvearrowright Y$ that are SOE (we assume OE for simplicity). We want to show the OE "preserves" the collection $\{H_\lambda\}_{\lambda\in\Lambda}$.

We can encode the information of the action $G \curvearrowright X$ into two things:

1. A measured groupoid $\mathcal{G}$ (a small category such that every morphism has an inverse). Objects: points in X. Morphisms (arrows): $x \to gx$.
2. A cocycle $\mathcal{G} \to G$ which assigns each arrow of $\mathcal{G}$ an element of G. This assignment is compatible with composition.

Preservation of subgroups

As $G \curvearrowright X$ and $G \curvearrowright Y$ are orbit equivalent, the groupoids from $G \curvearrowright X$ and from $G \curvearrowright Y$ are isomorphic, denoted $\mathcal{G}$.

However, we have two different cocycles, $\rho_1 : \mathcal{G} \to G$ from $G \curvearrowright X$ and $\rho_2 : \mathcal{G} \to G$ from $G \curvearrowright Y$.

Take $H \in \{H_\lambda\}_{\lambda \in \Lambda}$. Let $\mathcal{G}_1 = \rho_1^{-1}(H)$ be the collection of arrows of $\mathcal{G}$ whose ρ_1 image is in H. Is it true that there is a (possibly different) $H' \in \{H_\lambda\}_{\lambda \in \Lambda}$ such that $\rho_1^{-1}(H) = \rho_2^{-1}(H')$?

Kida proved that if G is a mapping class group, $\{H_\lambda\}_{\lambda \in \Lambda}$ is the collection of curve stabilizer subgroups. Then the answer is positive up to a countable partition of X, i.e. there is a countable partition $X = \sqcup X_j$ such that for each X_j, there exists $H_i \leqslant G$ such that $\rho_1^{-1}(H)|_{X_i} = \rho_2^{-1}(H_i)|_{X_i}$.

We say the collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ is invariant under the given OE if the above question is true up to a countable partition of X.

Definition: A countable group G is ME-superrigid if another countable group ME to G is virtually G.

Question (vague): are there natural examples of ME-superrigid groups obtained by "gluing" amenable groups together in a complicated pattern?

Attempt 1: Let G be obtained by taking amalgamations or HNN extension of amenable groups, i.e. G is a graph of group such that each vertex group is amenable.

Attempt 2: Let G be a complex of group such that each vertex group is amenable. (i.e. G acts on a cell-complex X with amenable cell-stabilizers).

Goal of this talk:

1. A general criterion guarantee vertex stabilizers are ME-invariants (in an appropriate sense) when X is "negatively curved" and the action of G on X is acylindrical.
2. A ME-superrigid result for most generalized Higman groups.

Higman groups

Recall that the Baumslag–Solitar group $BS(n,m) = \langle a, b \mid ab^n a^{-1} = b^m \rangle$
When $n = 1, m = 2$; $BS(1,2) = \langle a, b \mid aba^{-1} = b^2 \rangle$.
When $n = m = 1$, $BS(1,1) = \langle a, b \mid aba^{-1} = b \rangle$.

For each integer $k \geqslant 4$, Higman defined the following group:

$$\mathrm{Hig}_k = \langle a_1, \ldots, a_k | \{a_i a_{i+1} a_i^{-1} = a_{i+1}^2\}_{i \in \mathbb{Z}/k\mathbb{Z}} \rangle$$

When $k = 3$, the group is trivial.

1. Higman groups are the first examples of infinite finitely presented groups without any nontrivial finite quotient.
2. Higman groups play a key role in the construction of Grothendieck pairs (G, H) by Platonov and Tavgen' $(G = F_n \times F_n,\ H < G)$.
3. They are considered as potential examples for non-sofic groups (still open...)

Main results

Generalized Higman groups: Let $k \geqslant 4$, and let $\sigma = ((m_1, n_1), \ldots, (m_k, n_k))$ be a k-tuple of pairs of non-zero integers.

$$\mathrm{Hig}_\sigma = \langle a_1, \ldots, a_k | \{a_i a_{i+1}^{m_i} a_i^{-1} = a_{i+1}^{n_i}\}_{i \in \mathbb{Z}/k\mathbb{Z}} \rangle.$$

1. When $(m_i, n_i) = (1, 2)$ for all i, we recover the classical Higman groups;
2. when $(m_i, n_i) = (1, 1)$ for all i, we get right-angled Artin groups!

Theorem (Horbez-H. 2022)

Suppose $k \geqslant 5$ and $|m_i| \neq |n_i|$ for all i. Then Hig_σ is ME-superrigid, i.e. any countable group ME to Hig_σ is virtually Hig_σ.

Speculations: the theorem should still be true when $k = 4$. However, if $|m_i| = |n_i|$ for some i, then we might lose rigidity...

Let F_σ be the finite group consisting of all permutations of generators of Hig_σ that can be extended to a group automorphism. Let $\widehat{\mathrm{Hig}}_\sigma = \mathrm{Hig}_\sigma \rtimes F_\sigma$.

Corollary

Let $k \geqslant 5$, and let $\sigma = ((m_1, n_1), \ldots, (m_k, n_k))$ be a k-tuple of pairs of non-zero integers, with $|m_i| \neq |n_i|$ for every $i \in \{1, \ldots, k\}$.

1. *For every locally compact second countable group G and every lattice embedding $\alpha : \mathrm{Hig}_\sigma \to G$, there exists a continuous homomorphism $\pi : G \to \widehat{\mathrm{Hig}}_\sigma$ with compact kernel such that $\pi \circ \alpha = \mathrm{id}$*
2. *For every finite generating set S of Hig_σ, the inclusion $\mathrm{Hig}_\sigma \hookrightarrow \widehat{\mathrm{Hig}}_\sigma$ extends to an injective homomorphism $\mathrm{Aut}(\mathrm{Cay}(\mathrm{Hig}_\sigma, S)) \hookrightarrow \widehat{\mathrm{Hig}}_\sigma$.*

Corollary

(under the same assumption as before) Let $\mathrm{Hig}_\sigma \curvearrowright X$ be a free, ergodic, p.m.p. action on X. Let $\Gamma \curvearrowright Y$ be a free, ergodic, p.m.p. action on Y. If the cross-product von Neumann algebra of $\mathrm{Hig}_\sigma \curvearrowright X$ and $\Gamma \curvearrowright Y$ are isomorphic, then they are virtually conjugate.

Proof sketch

$$G = \langle a_1, \ldots, a_k | \{a_i a_{i+1}^{m_i} a_i^{-1} = a_{i+1}^{n_i}\}_{i \in \mathbb{Z}/k\mathbb{Z}} \rangle$$

A BS-subgroup of G is a conjugate of $\langle a_i, a_{i+1} \rangle$ for some i.

1. Prove BS-subgroups are ME-invariants in the sense explained before.
2. Show the intersection pattern of these BS-subgroups are rigid.

(Martin's curve graph analogue) Let Λ be a graph whose vertices are in 1-1 correspondence with BS-subgroups of G. Two vertices are adjacent iff the corresponding BS-subgroups have infinite intersection.

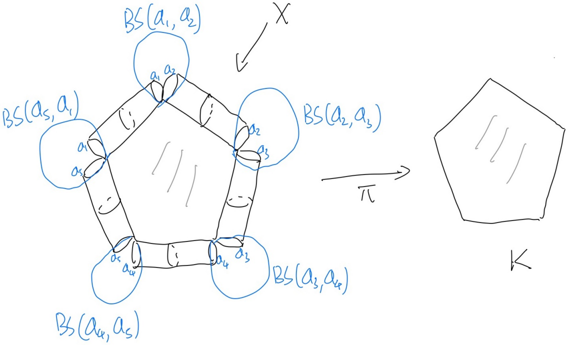
Theorem (Horbez-H.)

Suppose $k \geqslant 4$ and $|m_i| \neq |n_i|$ for any i. Then the natural map $G \to \mathrm{Aut}(\Lambda)$ has finite index image.

Rmk: The assumption $k \geqslant 5$ is used in Step 1.

An auxiliary complex for generalized Higman groups

Suppose $G = \langle a_1, \ldots, a_5 | \{a_i a_{i+1} a_i^{-1} = a_{i+1}^2\}_{i \in \mathbb{Z}/5\mathbb{Z}} \rangle$



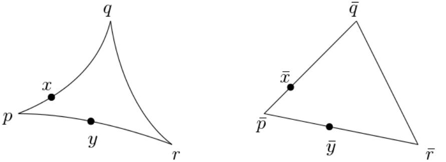
Note that $\pi_1 X = G$. Let $\widetilde{X}$ be the universal cover of X. We collapse each vertex space of $\widetilde{X}$ into a point, collapse each $[0,1] \times \mathbb{R}$ (which is a lift a cylinder in X) into the $[0,1]$ factor. The resulting space is denoted by $\hat{K}$.

1. $\hat{K}$ is a union of pentagons;
2. there is a map $\widetilde{X} \to \hat{K}$.

Negative curvature of $\hat{K}$

Observation: If we metric each pentagon of $\hat{K}$ as a right-angled regular pentagon in the hyperbolic plane, then $\hat{K}$ is CAT(-1).

A geodesic metric space X is $CAT(-1)$ if triangles in X are thinner than those in the hyperbolic plane.



$d(x,y) \leq d(\bar{x},\bar{y})$

A general criterion for invariance of vertex groups

Theorem

[Horbez-H.] Let X be a connected $\mathrm{CAT}(-1)$ piecewise hyperbolic polyhedral complex with countably many cells in finitely many isometry types. Let G be a torsion-free countable group acting by cellular isometries on X. Assume that

1) *(Vertex stabilizers). The stabilizer of every vertex of X is amenable.*
2) *(Edge stabilizers). Edge stabilizers for the G-action on X are of infinite index in the incident vertex stabilizers.*
3) *(Weak acylindricity). The G-action on X is weakly acylindrical.*
4) *(Non-isolation of amenable vertex stabilizers). For each vertex v, there exists an infinite subgroup of $\mathrm{Stab}_G(v)$ which fixes two distinct vertices of X which are different from v.*

Then the collection of vertex group of G are SOE invariants in the sense explained before.

Comments on assumptions

Consider $G = \mathbb{Z}^2 * \mathbb{Z}^2$. Then G acts on the Bass-Serre tree with amenable stabilizers. Any tree is CAT(-1).

The vertex groups can not be SOE invariants in the sense mentioned before, as G is orbit equivalent to $\mathbb{Z} * \mathbb{Z}$.

Thus we want the vertex groups to be non-isolated (avoid relative hyperbolic situation).

Let H be a group acting on a metric space Z. The H-action on Z is said to be *weakly acylindrical* if there exist $L > 0, N > 0$ such that for any two points $x, y \in Z$ with $d(x, y) \geqslant L$, the common stabilizer of x and y has cardinality at most N.

A modified Adam-Kida style argument

Let $A \leqslant G$ be an maximal infinite amenable subgroup which is not isolated. Need to show: A fixes a vertex of X.

Strategy: promote invariant probability measure into fixed points.
Let $K = X \cup \partial_\infty X$. $A \curvearrowright K$. If K is compact, then we have a measure μ invariant under A-action.

Case 1: If μ is supported on X. Select using countability.
Case 2: Suppose μ is supported on $\partial_\infty X$.

1. If the support has $\geqslant 3$ pts. There is a Borel G-equivariant barycentric map from three distinct points of $\partial_\infty X$ to points of X. This reduces to Case 1.
2. If every A-invariant μ has support at most 2 pts. Then the weak acylindricity implies that A is isolated, contradiction.

Thank you!