Boundary rigidity of Gromov hyperbolic spaces

Hao Liang

Shanghai
August 2022

(This is joint work with Qingshan Zhou.)

X: δ-hyperbolic space ($[x, y] \subset N_\delta([y, z] \cup [x, z])$)

Gromov boundary of X:
$\partial X = \{[\gamma] \mid \gamma : [0, \infty) \to X$ is a geodesic ray with $\gamma(0) = o\}$,

A quasi isometry $f : X \to X$ induces $\partial f : \partial X \to \partial X$.

(This is joint work with Qingshan Zhou.)

X: δ-hyperbolic space ($[x, y] \subset N_\delta([y, z] \cup [x, z])$)

Gromov boundary of X:
$\partial X = \{[\gamma] \mid \gamma : [0, \infty) \to X$ is a geodesic ray with $\gamma(0) = o\}$,

A quasi isometry $f : X \to X$ induces $\partial f : \partial X \to \partial X$.

Given $f : X \to X$, we call $\sup_{x \in X} d(x, f(x))$ the *displacement* of f.

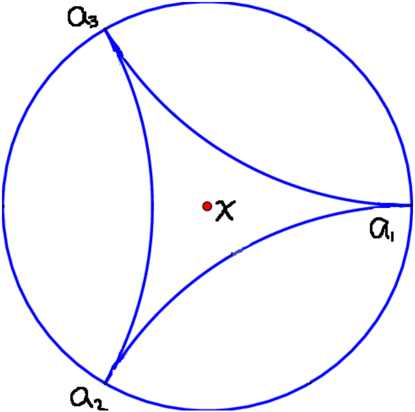
Definition (Boundary rigid)

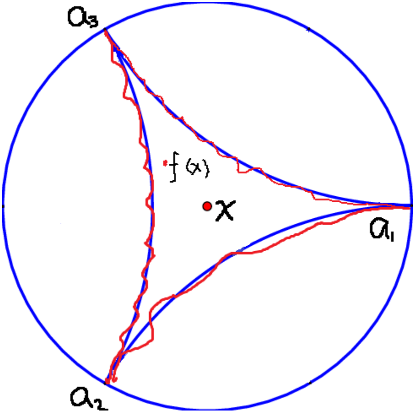
(X, d) is *boundary rigid* : For any quasi-isometry $f : X \to X$, if $\partial f : \partial X \to \partial X$ is the identity map, then f has finite displacement.

Example 1: $X = \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$.
∂f fixes the boundary of $\mathbb{R}$, but f has infinite displacement.
$\mathbb{R}$ is not boundary rigid.

Example 1: $X = \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$.
∂f fixes the boundary of $\mathbb{R}$, but f has infinite displacement.
$\mathbb{R}$ is not boundary rigid.

Example 2: $X = \mathbb{H}$.
Let $f : \mathbb{H} \to \mathbb{H}$ be q.i. so that $\partial f = 1_{\partial \mathbb{H}}$.
Given $x \in \mathbb{H}$, there exists $a_1, a_2, a_3 \in \partial \mathbb{H}$ s.t. $d(x, [a_i, a_j]) \leq D$ for some uniform D.


a3
x
a1
a2


a3
f(x)
x
a1
a2

Example 1: $X = \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$.
∂f fixes the boundary of $\mathbb{R}$, but f has infinite displacement.
$\mathbb{R}$ is not boundary rigid.

Example 2: $X = \mathbb{H}$.
Let $f : \mathbb{H} \to \mathbb{H}$ be q.i. so that $\partial f = 1_{\partial \mathbb{H}}$.
Given $x \in \mathbb{H}$, there exists $a_1, a_2, a_3 \in \partial \mathbb{H}$ s.t. $d(x, [a_i, a_j]) \leq D$ for some uniform D.
Then $d(f(x), f([a_i, a_j])) \leq D'$. Hence $d(x, f(x)) \leq D''$.
$\mathbb{H}$ is boundary rigid.

Let $\mathrm{QI}(X) = \{f \mid f : X \to X \text{ is a quasi-isometry}\}/\sim$, where $f \sim g$ iff $\sup_{x \in X} d(f(x), g(x))$ is finite.
If $f' \sim f$, then $\partial f = \partial f'$. Therefore, there is a homomorphism $\partial : \mathrm{QI}(X) \to \mathrm{Homeo}(\partial X)$ which sends f to ∂f.

Let $\mathrm{QI}(X) = \{f \mid f : X \to X \text{ is a quasi-isometry}\}/\sim$, where $f \sim g$ iff $\sup_{x \in X} d(f(x), g(x))$ is finite.
If $f' \sim f$, then $\partial f = \partial f'$. Therefore, there is a homomorphism $\partial : \mathrm{QI}(X) \to \mathrm{Homeo}(\partial X)$ which sends f to ∂f.

Definition (Boundary rigid 2)

X is *boundary rigid* if $\partial : \mathrm{QI}(X) \to \mathrm{Homeo}(\partial X)$ is injective.

Let $\mathrm{QI}(X) = \{f \mid f : X \to X \text{ is a quasi-isometry}\}/\sim$, where $f \sim g$ iff $\sup_{x \in X} d(f(x), g(x))$ is finite.
If $f' \sim f$, then $\partial f = \partial f'$. Therefore, there is a homomorphism $\partial : \mathrm{QI}(X) \to \mathrm{Homeo}(\partial X)$ which sends f to ∂f.

Definition (Boundary rigid 2)

X is *boundary rigid* if $\partial : \mathrm{QI}(X) \to \mathrm{Homeo}(\partial X)$ is injective.

Fact (boundary rigidity is q.i. invariant)

If X is boundary rigid and Y is quasi isometric to X, then Y is also boundary rigid.

Definition

A geodesic space X has an *pole* if there exists $L \geq 0$ and $o \in \bar{X} = X \cup \partial X$ s.t. each point of X lies in the L-neighborhood of some geodesic connecting o to some point $v \in \partial X$. We call o a pole of X.

Roughly star-like; visual

Definition

A geodesic space X has an *pole* if there exists $L \geq 0$ and $o \in \bar{X} = X \cup \partial X$ s.t. each point of X lies in the L-neighborhood of some geodesic connecting o to some point $v \in \partial X$. We call o a pole of X.

Roughly star-like; visual

Examples of Gromov hyperbolic spaces having a pole:

1. infinite hyperbolic groups,
2. Gromov hyperbolic domains in $\mathbb{R}^n$,
3. Gromov hyperbolic manifolds with a pole,
4. hyperbolic fillings.

Example (Space without a pole):

$x \in \mathbb{H}$. Let

$$X = \mathbb{H} \cup \left(\bigcup_{n \in \mathbb{N}} [0, n] \right) / 0 \sim x$$

X has no pole.

Example (Space without a pole):
$x \in \mathbb{H}$. Let

$$X = \mathbb{H} \cup (\bigcup_{n \in \mathbb{N}} [0, n])/0 \sim x$$

X has no pole.

$f_n : [0, n] \to [0, n]$

$$f_n(x) = \begin{cases} 2x, & \text{if } x \in [0, n/3], \\ 2n/3 + \frac{1}{2}(x - n/3), & \text{if } x \in [n/3, n]. \end{cases}$$

Displacement of f_n goes to ∞

Let $f : X \to X$ be defined by $1_{\mathbb{H}}$ and $\{f_n\}$
Then f is a $(2, 0)$-quasi-iso and has infinite displacement.
X is not boundary rigid.

Example (Space without a pole):
$x \in \mathbb{H}$. Let

$$X = \mathbb{H} \cup (\bigcup_{n\in\mathbb{N}} [0, n])/0 \sim x$$

X has no pole.

$f_n : [0, n] \to [0, n]$

$$f_n(x) = \begin{cases} 2x, & \text{if } x \in [0, n/3], \\ 2n/3 + \frac{1}{2}(x - n/3), & \text{if } x \in [n/3, n]. \end{cases}$$

Displacement of f_n goes to ∞

Let $f : X \to X$ be defined by $1_{\mathbb{H}}$ and $\{f_n\}$
Then f is a $(2, 0)$-quasi-iso and has infinite displacement.
X is not boundary rigid.

Question: Is having a pole a necessary condition for boundary rigidity?

Theorem (L.-Zhou)

Suppose that X is a proper geodesic Gromov hyperbolic space with a pole. Then the following conditions are equivalent:

1. X is boundary rigid.
2. ∂X is uniformly perfect.

Theorem (L.-Zhou)

Suppose that X is a proper geodesic Gromov hyperbolic space with a pole. Then the following conditions are equivalent:

1. X is boundary rigid.
2. ∂X is uniformly perfect.

(2) $\Rightarrow$ (1) proved by Bonk, Heinonen and Koskela.

Zhou has a quantitative version motivated by the Teichmuller displacement problem: D (domain of $\mathbb{R}^n$) $f : D \to D$ quasiconformal and f fixes ∂D. Find smallest upper bound for the displacement of f.

$x \in \mathbb{H}$. Let

$$X = \mathbb{H} \cup (\bigcup_{n \in \mathbb{N}} [0, n])/0 \sim x$$

$\partial X = \partial \mathbb{H} = S^1$, which is uniformly perfect. However, X is not boundary rigid.

Every Gromov hyperbolic space with a uniformly perfect boundary can be embedded into a Gromov hyperbolic space with the same boundary that is not boundary rigid.

Example (non-connected):

Let $x_1 = 1$, $x_n = n(x_1 + \cdots + x_{n-1})$, $n \in \mathbb{Z}$
Let $X = \{\pm x_n \mid n \in \mathbb{Z}\} \subset \mathbb{R}$.

Let $f : X \to X$ be a (K, C)-quasi-isometry such that its induced map ∂f fixes ∂X. There exists n_0 depending only on K and C, such that if $m > n \geq n_0$, then we have $|x_m - x_{m+1}| > K|x_n - x_{n+1}| + C$.

One can show that f has to fix x_m for all $m > n_0$. Hence X is boundary rigid but ∂X is not uniformly perfect.

Definition

(X, d) is *uniformly perfect*: There exists $S \geq 1$ and $r_0 > 0$ s.t. for each $x \in X$ and every $0 < r \leq r_0$, we have

$$r/S < d(x, y) \leq r$$

for some $y \in X$.

Examples:

1. connected spaces,
2. many disconnected fractals such as the Cantor set, Julia sets and the limit sets of nonelementary, finitely generated Kleinian groups.
3. boundary of non-elementary hyperbolic groups.

Corollary

X is a non-compact complete Riemannian manifold or a uniform graph which has bounded local geometry. If X is Gromov hyperbolic and has a pole, then the following are equivalent:

1. *X is boundary rigid.*
2. *∂X is uniformly perfect.*
3. *X has positive Cheeger constant.*

The equivalence of (2) and (3) is established by Martínez-Pérez and Rodríguez.

Corollary

X is a non-compact complete Riemannian manifold or a uniform graph which has bounded local geometry. If X is Gromov hyperbolic and has a pole, then the following are equivalent:

1. *X is boundary rigid.*
2. *∂X is uniformly perfect.*
3. *X has positive Cheeger constant.*

The equivalence of (2) and (3) is established by Martínez-Pérez and Rodríguez. A graph: $\Gamma = (V(\Gamma), E(\Gamma))$; d_Γ: length metric. For $A \subset V(\Gamma)$, let $\partial A = \{v \in V(\Gamma) \mid d_\Gamma(v, A) = 1\}$.
Cheeger constant of a graph:

$$h(\Gamma) = \inf_{A \subset V(\Gamma), 0 < |A| < \infty} \frac{|\partial A|}{|A|}$$

"geodesically rich"(Shchur) implies "boundary rigid".

Definition

(X, d) is *geodesically rich* if there are constants r_0, r_1, r_2 s.t. for $p, q \in X$ with $d(p, q) > r_0$, there exists a bi-infinite geodesic γ such that $d(p, \gamma) < r_1$ and $|d(q, \gamma) - d(q, p)| < r_2$.

Example: $\mathbb{R}$ no. $\mathbb{H}$ yes.

"geodesically rich"(Shchur) implies "boundary rigid".

Definition

(X, d) is *geodesically rich* if there are constants r_0, r_1, r_2 s.t. for $p, q \in X$ with $d(p, q) > r_0$, there exists a bi-infinite geodesic γ such that $d(p, \gamma) < r_1$ and $|d(q, \gamma) - d(q, p)| < r_2$.

Example: $\mathbb{R}$ no. $\mathbb{H}$ yes.

Corollary

Let X be a proper geodesic Gromov hyperbolic space with a pole. Then the following are equivalent:

1. *X is boundary rigid.*
2. *∂X is uniform perfect.*
3. *X is geodesically rich.*

Question(Shchur): Can a hyperbolic space always be isometrically embedded into a geodesically rich hyperbolic space with an isomorphic boundary?

Question(Shchur): Can a hyperbolic space always be isometrically embedded into a geodesically rich hyperbolic space with an isomorphic boundary?

No. $\mathbb{R}$: since all the bi-infinite geodesics of a hyperbolic space whose boundary consists of two points fellow travel.

Proposition

Let X be a proper geodesic Gromov hyperbolic space. If X quasi-isometrically embeds into a geodesically rich Gromov hyperbolic space Y inducing a bijection between ∂X and ∂Y, then X is quasi-isometric to Y and X is boundary rigid.

Idea of the proof: Uniformly perfect $\Rightarrow$ boundary rigid:

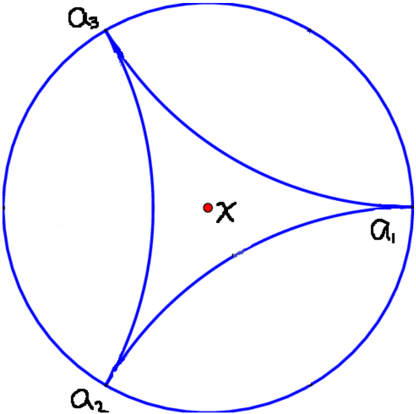
Definition

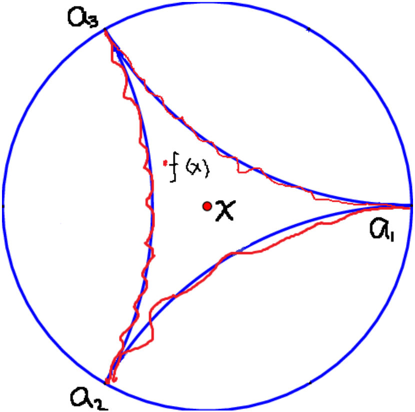
Let $K \geq 1$, $C \geq 0$ and $\rho \geq 0$. A point $x \in X$ is called a (K, C, ρ)-*quasi-centroid* of a triple of points $\{x_1, x_2, x_3\} \subset \bar{X}$ if there are (K, C)-quasi-geodesics $[x_1, x_2]$, $[x_2, x_3]$ and $[x_1, x_3]$ such that x is within ρ of each of them.

Let $\Delta_{(K,C,\rho)}(\{x_1, x_2, x_3\})$ be the set of all (K, C, ρ)-quasi-centroids of $\{x_1, x_2, x_3\}$.

Fact (Meyer, Bourdon-Kleiner): ∂X is uniformly perfect $\Leftrightarrow$ $\bigcup_{x_1,x_2,x_3\in\partial} \Delta_{(1,0,\rho)}(\{x_1, x_2, x_3\}) = X$ for some ρ.

Fact (Behrstock-Minsky) : The diameter of $\Delta_{(K,C,\rho)}(\{x_1, x_2, x_3\})$ is bounded by $D(K, C, \rho, \delta)$.


a3
x
a1
a2


a_3
f(x)
x
a_1
a_2

boundary rigid $\Rightarrow$ uniformly perfect:

Suppose ∂X is not uniformly perfect.

Fact (Meyer, Bourdon-Kleiner) implies: there exist $B_i \subset X - \bigcup_{x_1,x_2,x_3\in\partial} \Delta_{(1,0,\rho)}(\{x_1,x_2,x_3\})$ s.t. $\mathrm{diam}(B_i) \to \infty$ and $B_i \cap B_j =$.

Goal:

1. B_i q.i. to $[0, n]$
2. B_i is "connected to" $X - \bigcup_{x_1,x_2,x_3\in\partial} \Delta_{(1,0,\rho)}(\{x_1,x_2,x_3\})$ in a "simple" way.

Then construct a q.i with infinite displacement as in the previous example.

Thank you!