Non-Abelian Hodge correspondence and certain Hitchin fibers

Geometric Group theory Graduate Summer School
@ Fudan University, Shanghai

joint work with Song Dai (Tianjin University)

Qiongling Li
Chern Institute of Mathematics
Nankai University

2022.8.18

Domination between representations

Let S be a closed oriented surface of $g \geq 2$.

Consider a representation $\rho : \pi_1(S) \to PSL(n, \mathbb{C})$.

Let $N = SL(n,\mathbb{C}) / SU(n) \curvearrowleft PSL(n,\mathbb{C})$

equipped with an inv Riem metric induced by the trace form on $\mathfrak{sl}(n,\mathbb{C})$.

Let d_N denote induce distance fcn.

- The translation length spectrum $\ell_\rho : \pi_1(S) \to \mathbb{R}^{>0}$ is $\ell_\rho(\gamma) := \min_{x \in N} d_N(x, \rho(\gamma)x)$.

- The entropy of ρ is
$$h_\rho := \limsup_{R \to \infty} \frac{\log \left\{ \# \{ \gamma \in \pi_1(S) \mid \ell_\rho(\gamma) \leq R \} \right\}}{R}$$

- Call $\rho < j$ if $\exists\, \lambda < 1$ s.t. $\ell_\rho(\gamma) < \lambda\, \ell_j(\gamma)$ $\forall\, \gamma \overset{\neq \mathrm{id}}{\in} \pi_1(S)$.

Fuchsian representations

For a hyperbolic surface (S, g),

let j be the uniformizing rep

$$j: \pi_1(S) \to PSL_2\mathbb{R}.$$

(realizing $\pi_1(S)$ as isometries of $\mathbb{H}^2$).

Such rep j are called Fuchsian

($\Leftarrow$) discrete and faithful.)

Fact: • $h_j \equiv 1$.

• Any two Fuchsian reps cannot dominate each other.

Thm 1 (Deroin – Tholozan IMRN 16')

Given a representation $\rho: \pi_1(S) \to PSL(2,\mathbb{C})$,

$\exists$ a Fuchsian rep $j: \pi_1(S) \to PSL(2,\mathbb{R})$ s.t.

$$\rho < j \quad \text{unless } \rho \text{ is Fuchsian.}$$

As a result, $h_\rho > 1$ unless ρ is Fuchsian.

In fact, they show much more.

Restrict ρ to be reductive.

" $\rho: \pi_1(S) \to G$, $\mathrm{Ad} \circ \rho: \pi_1(S) \to GL(\mathfrak{g})$ is completely reducible. "

Associate q_2 to a rep $\rho: \pi_1(S) \to PSL(2,\mathbb{C})$

Given a R.S str X on S,
for any red rep $\rho: \pi_1(S) \to PSL_2\mathbb{C}$,
one can associate $q_2 \in H^0(X, K_X^2)$.

$q_2(z)dz^2$ ($q_2(z)$ holo^m).

- By Donaldson's thm,
  $\exists$ a ρ-equiv harmonic map
  $f: \boxed{\widetilde{X}} \to \mathbb{H}^3 = SL(2,\mathbb{C})/SU(2)$.

  The quadratic diff $:=$ Hopf diff of f.
  i.e. $(2,0)$-part of $f^* g_{\mathbb{H}^3}$.
  holo^m follows from harmonicity of f.

  $\mathbb{H}^3 \cong$ space of pos. def Herm matrices of $\det = 1$,

  $$\mathrm{Hopf}(f) := \mathrm{tr}(\underbrace{f^{-1}\partial f}_{\in \Omega^{1,0}(X, \quad)} \otimes f^{-1}\partial f)$$

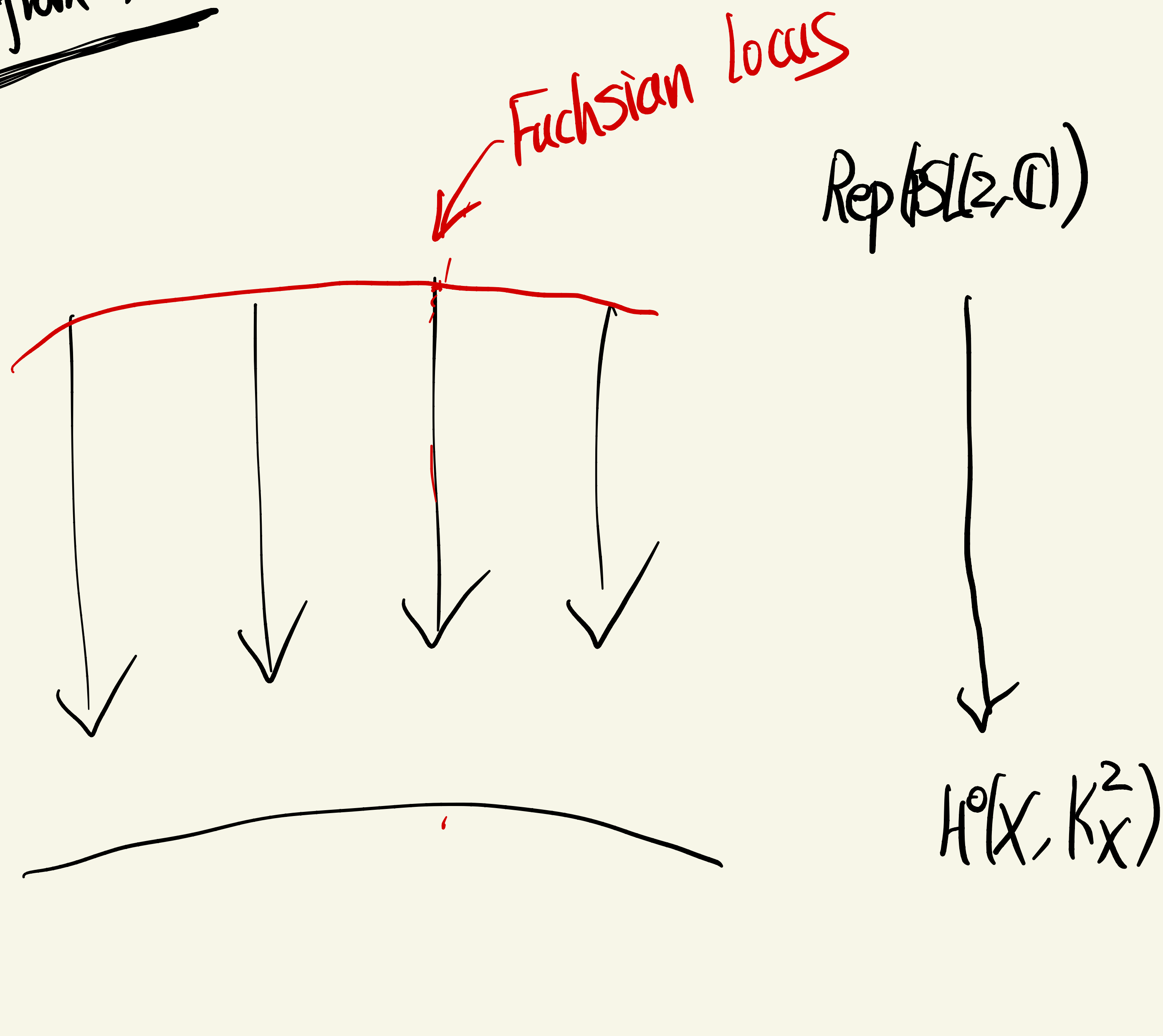
- Conversely, given (X, q_2),
  by the result of Wolf, Hitchin,
  $\exists!$ a Fuchsian rep $j_{q_2}: \pi_1(S) \to PSL_2\mathbb{R}$ s.t its associated diff is q_2.

Thm 2 (Deroin - Tholozan, 16)

Given (X, q_2), all the reductive representations $\rho: \pi_1(S) \to PSL(2,\mathbb{C})$ associated to q_2 is dominated by j_{q_2}.

$$\mathrm{Rep}(PSL(2,\mathbb{C})) = \mathrm{Hom}^{red}(\pi_1(S), PSL_2\mathbb{C}) / PSL_2\mathbb{C}.$$

View from X.



Literature on Deroin-Tholozan's thms (1+2).

Note that Thm 2 $\Longrightarrow$ Thm 1
$\not\Longleftarrow$

On Thm 2, noncompact surfaces
N. Sagman JDG 2022'.

On Thm 1, surfaces with bdry. (keep bdry length).
S. Gupta - Weixu Su. ArXiv 2020'.

Our goal: Consider $PSL(n, \mathbb{C})$-rep.
Ask right formulation of $PSL(n,\mathbb{C})$-case?

Thm (Corlette, $n \geq 3$)

Given a reductive rep $\rho : \pi_1(S) \longrightarrow PSL(n,\mathbb{C})$,
$\exists$ a ρ-equiv harmonic map $f : \tilde{X} \longrightarrow SL(n,\mathbb{C})/SU(n)$.

Conversely, the existence of harmonic maps implies reductivity.

Identify $SL(n,\mathbb{C})/SU(n)$ with space of pos. def Herm matrices of $\det = 1$.

We associate a tuple of holo^m diff
$q_2, q_3, \dots, q_n$ $\quad (q_2 = \mathrm{Hopf}(f))$.

where $q_i = \mathrm{tr}(\underbrace{f^{-1}\partial f \otimes f^{-1}\partial f \otimes \dots \otimes f^{-1}\partial f}_{i \text{ times}})$.

- $f^{-1}\partial f$ is holo^m section of a holo^m v.b
  $\Leftarrow$ harmonicity of f.

n-Fuchsian reps

There is a unique irreducible rep

$$\tau_n : PSL(2,\mathbb{C}) \longrightarrow PSL(n,\mathbb{C})$$

$$A \longmapsto \text{the induced action on the basis}$$

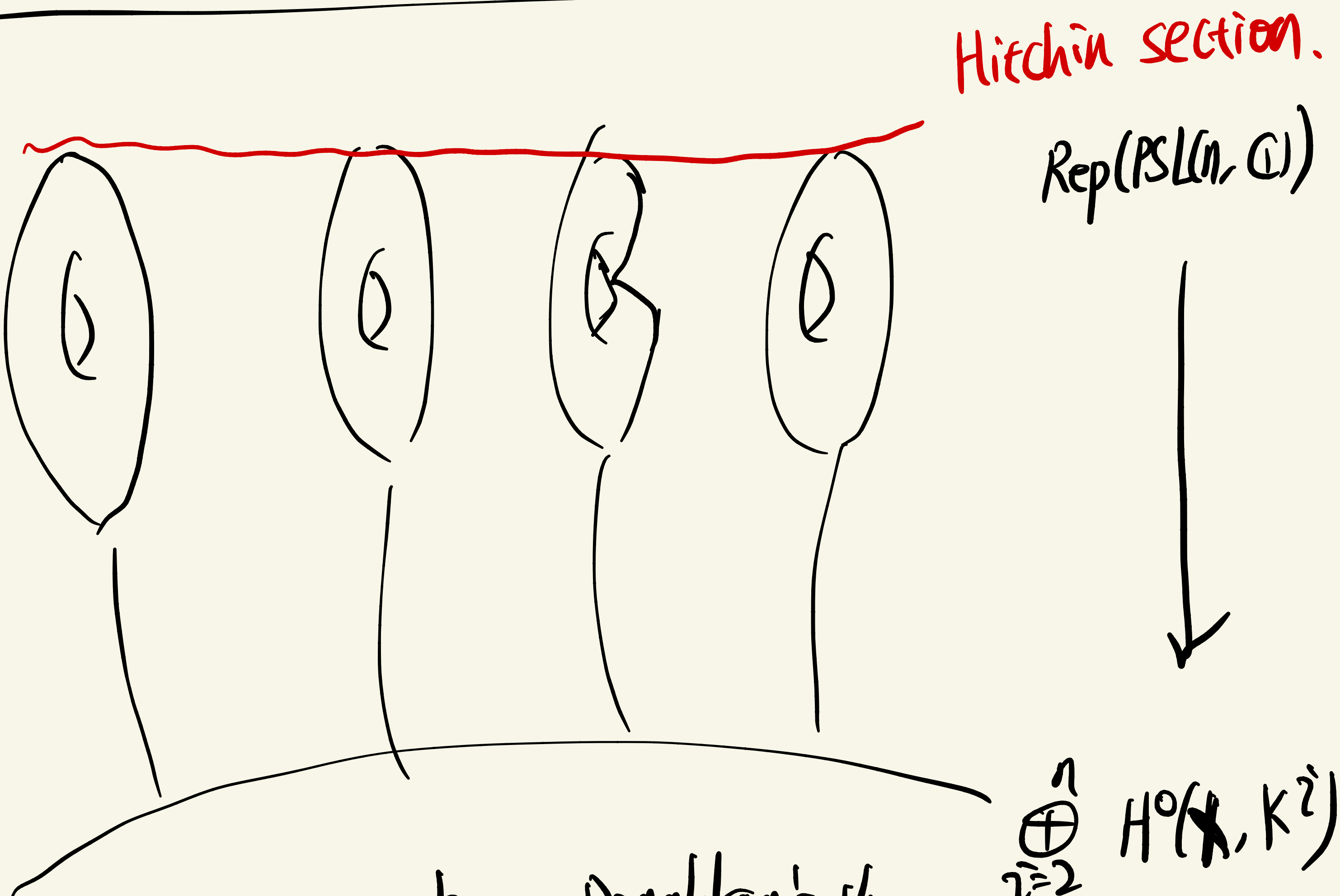
$$A \curvearrowright \begin{pmatrix} X \\ Y \end{pmatrix} \qquad \{X^{n-1}, X^{n-2}Y, \dots, Y^{n-1}\}.$$

Given a Fuchsian rep $j : \pi_1(S) \to PSL_2\mathbb{R}$,

obtain $\tau_n \circ j : \pi_1(S) \to PSL_n\mathbb{R}$.

called <u>n-Fuchsian rep</u>.

Fibration of $Rep(PSL(n,\mathbb{C}))$ depending on X



- Fibration generalizes Donaldson's thm.
- Conversely, giving $(q_2, q_3, \dots, q_n)$,
  Hitchin defines a section,

Hitchin. rep lies in $PSL(n,\mathbb{R})$.

- Hitchin reps are exactly continuous deformations
  of n-Fuchsian locus in $Rep(PSL(n,\mathbb{R}))$.

Relation to the non-Abelian Hodge correspondence

NAH.

The NAH is 1-1 bijection between

$$\rho \in \mathrm{Rep}(PSL(n,\mathbb{C})) \dashleftarrow\dashrightarrow \{\text{Higgs bundles}\}/\sim$$

topological   holo^m data.

Corlette, Donaldson

ρ-equiv harmonic maps

$$f: \tilde{X} \to SL(n,\mathbb{C})/SU(n).$$

__Defn__. A Higgs bundle over X is a pair $\underline{(E,\Phi)}$.

— E is rk n holo^m v.b.

— $\Phi \in H^0(X, \mathrm{End}(E) \otimes K)$.

$$\left\{ \begin{array}{l} \text{Equivariant harmonic maps} \\ f : \tilde{X} \to SL(n,\mathbb{C})/SU(n) \end{array} \right\} \longrightarrow \begin{array}{l} \text{Higgs bundles over } X. \\ \left(f^*P \times_\iota \mathbb{C}^n ,\ f^{-1}\partial f \right) \end{array}$$

holom str arising from the (0,1)-part of the canonical connection on P.

$$\begin{tikzcd} f^*P \arrow[d] & P = SL(n,\mathbb{C}) \arrow[d] \\ \tilde{X} \arrow[r, "f"] & SL(n,\mathbb{C})/SU(n) \end{tikzcd}$$

is a principal $SU(n)$-bundle P.

Polystable Higgs bundles $\longrightarrow$ equivariant harmonic maps $f: \tilde{X} \to SL(n,\mathbb{C}) / SU(n)$

(E, Φ)

Hitchin, Simpson

parallel transport H to a single fiber.

$\exists H$ s.t

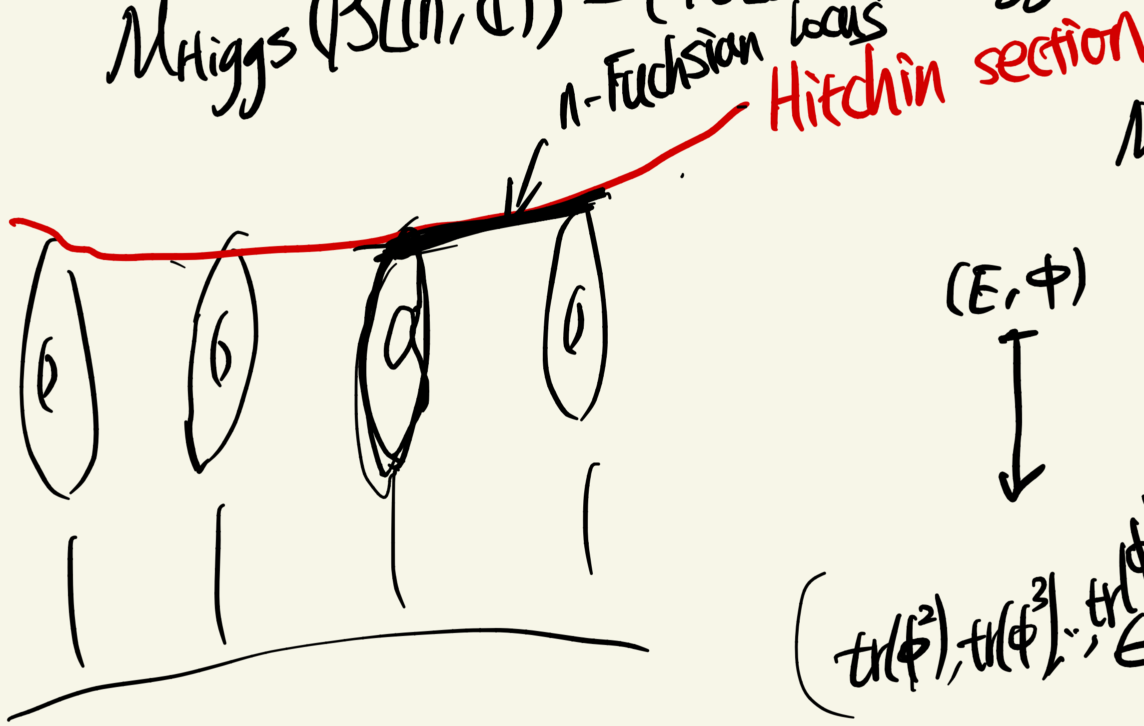
$$F(\nabla^H) + [\Phi, \Phi^{*H}] = 0$$

(Hitchin eqn)

$(\Leftrightarrow)$ $D = \nabla^H + \Phi + \Phi^{*H}$ is flat

Hitchin fibration.

$$\mathcal{M}_{Higgs}(PSL(n,\mathbb{C})) = \{PSL(n,\mathbb{C})\text{-Higgs bundles over } X\}^{\text{polystable}} / \sim$$



$$\mathcal{M}_{Higgs}(PSL(n,\mathbb{C}))$$

$$(E,\phi) \longmapsto \left(tr(\phi^2), tr(\phi^3), \ldots, tr(\phi^n)\right) \in \bigoplus_{i=2}^{n} H^0(X, K^i)$$

- geometric langlands program
  mirror symmetry.

Generalization of Thm 2.

<u>Conjecture</u> (Deroin-Tholozan, 16')

Is Hitchin section maximal in its Hitchin fiber in terms of length spectrum?

If not, <u>how about certain fibers?</u>

Generalization of Thm 1.

<u>Q1.</u> Given a rep $\rho: \pi_1(S) \longrightarrow PSL(n, \mathbb{C})$, does there exist a n-Fuchsian rep $\tau_n \circ j$ s.t. $\rho < \tau_n \circ j$?

This is not true.

- In fact, Hitchin rep are as large as possible.
- Moreover, Potrie-Sambarino. Invent. Math 17') showed that for Hitchin reps

$$h_\rho \leq h_{\tau_n \circ j} = \frac{1}{\sqrt{C_n}}, \quad \underline{C_n = \frac{6}{n(n^2-1)}}.$$

rigidity holds.

Generalization of Thm 1.

Q2. Given a rep $\rho: \pi_1(S) \to PSL(n, \mathbb{C})$, does there exist a Hitchin rep
$$\sigma: \pi_1(S) \longrightarrow PSL(n, \mathbb{R})$$
s.t. $\rho < \sigma$.

- For closed surface, unknown.
- For punctured surfaces, Borman - Gupta (ongoing work).

Thm (Dai-L., PLMS 2022')

For any rep ρ in the Hitchin fiber of a n-Fuchsian rep $\tau_n \circ j$,

then $\rho < \tau_n \circ j$ unless $\rho = \tau_n \circ j$.

As a result, $h_\rho > h(\tau_n \circ j) = \frac{1}{\sqrt{c_n}}$.

Rmk: $n=2$ case, any rep lies in a fiber of a Fuchsian rep.

$n>2$ case, this is not true.

Idea of proof:

Recall the proof in Deroin - Tholozan's thm.

Let K be the induced curvature of pullback metric.

Bochner formula for H = holo^m energy density

$$\boxed{\Delta \log H = -2K\left(H - \frac{\|q_2\|^2}{H}\right) - 2}$$

the only variant.

$\boxed{K_{f_j} \equiv -1.}$

$\boxed{K_{f_\rho} \leq -1.}$

If we have $K_1 \leq K_2 < 0.$

$\Rightarrow H_1 \leq H_2$ max principle.

$\Rightarrow$ Energy density$_{(1,1)}$ $\leq$ Energy density$_{(1,1)}$.

- For $f_{[n \circ j]}$, $K = K^N_{f_{[n \circ j]}}$ the sectional curvature
totally geodesic. $= -C_n$.

- For f_ρ, $K \leq K^N_{f_\rho}$

$SL(n,\mathbb{C}) / SU(n)$. has curvature ≤ 0.

Add missing part: $K^N_{f_\rho} \leq -C_n = K_{f_{\tau_{n,0}}}$.

If ρ and $\tau_{n,0}$ are the same Hitchin fiber.

Consider

$$K(A)=\frac{|[A,A^*]|^2}{|A|^4-|\langle A,A^*\rangle|^2}.$$

§

K^{N}_{fe}.

<u>Lemma</u>. Suppose A has eigenvalues of the form $t(n-1, n-3, \ldots, 1-n)$ $(t\in\mathbb{C})$, then

$$K(A)\geq C_n.$$

<u>Cor</u>. If A is nilpotent, $K(A)\geq C_n$. recovering a GIT thm by L. Ness and Schmit-Vilonen.

$\square$

In fact, the Higgs bundle (E, Φ) corresponding to $\tau_n \circ j$ is of the form

$$E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{\frac{3-n}{2}} \oplus K^{\frac{1-n}{2}}.$$

$$\Phi = \begin{pmatrix} 0 & \sqrt{r_1}\, q_2 & & & \\ \sqrt{r_1} & 0 & \sqrt{r_2}\, q_2 & & \\ & \sqrt{r_2} & 0 & \ddots & \\ & & \ddots & \ddots & \sqrt{r_{n-1}}\, q_2 \\ & & & \sqrt{r_{n-1}} & 0 \end{pmatrix}$$

$$: E \longrightarrow E \otimes K.$$

where $r_i = i(n-i)$