SPECIAL CUBE COMPLEXES AND 3-MANIFOLD GROUPS

PART III

Yi Liu (BICMR)

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- Some background

Wise: word hyp. + QVH ⇒ VS

hyp. 3 mfd ⇒ VF, VHaken

Agol: word-hyp + CAT(0) cubulation (cocpcE) ⇒ QVH

LERF (using tameness)

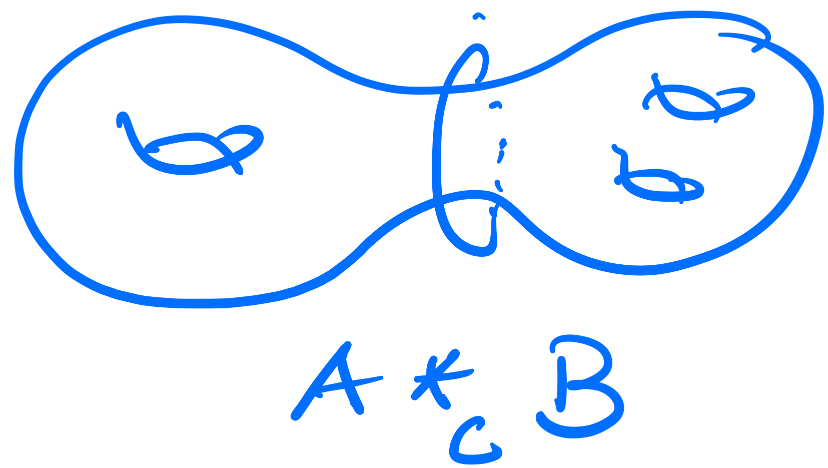
DEFINITION. QVH is the smallest class of groups such that

- $1 \in QVH$;
- C q.c. in A and B.
  $A, B \in QVH \Rightarrow A *_C B \in QVH$;
- C q.c. in A.
  $A \in QVH \Rightarrow A*_C \in QVH$;
- $A' \leq_{f.i.} A$
  $A' \in QVH \Rightarrow A \in QVH$.

EXAMPLE:

$\mathbb{Z} \in QVH$, $F_n \in QVH$,

surface groups $\in QVH$.



Motivating Agol's proof:

$$\underset{\text{word hyp.}}{G} \overset{\text{cocpct}}{\curvearrowright} \underset{\substack{\text{CAT(0)}\\ \text{cub. cplx}}}{X}$$

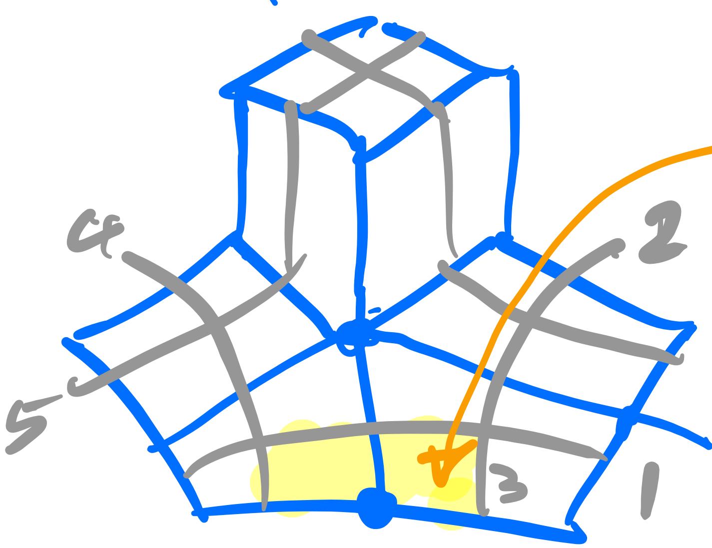
$$\Downarrow ?$$

$$\dot{G} \leq_{f.i.} G$$

$$\dot{G} \curvearrowright X$$
special

· If we already have a sp. cub. cplx

$\mathcal{V}$



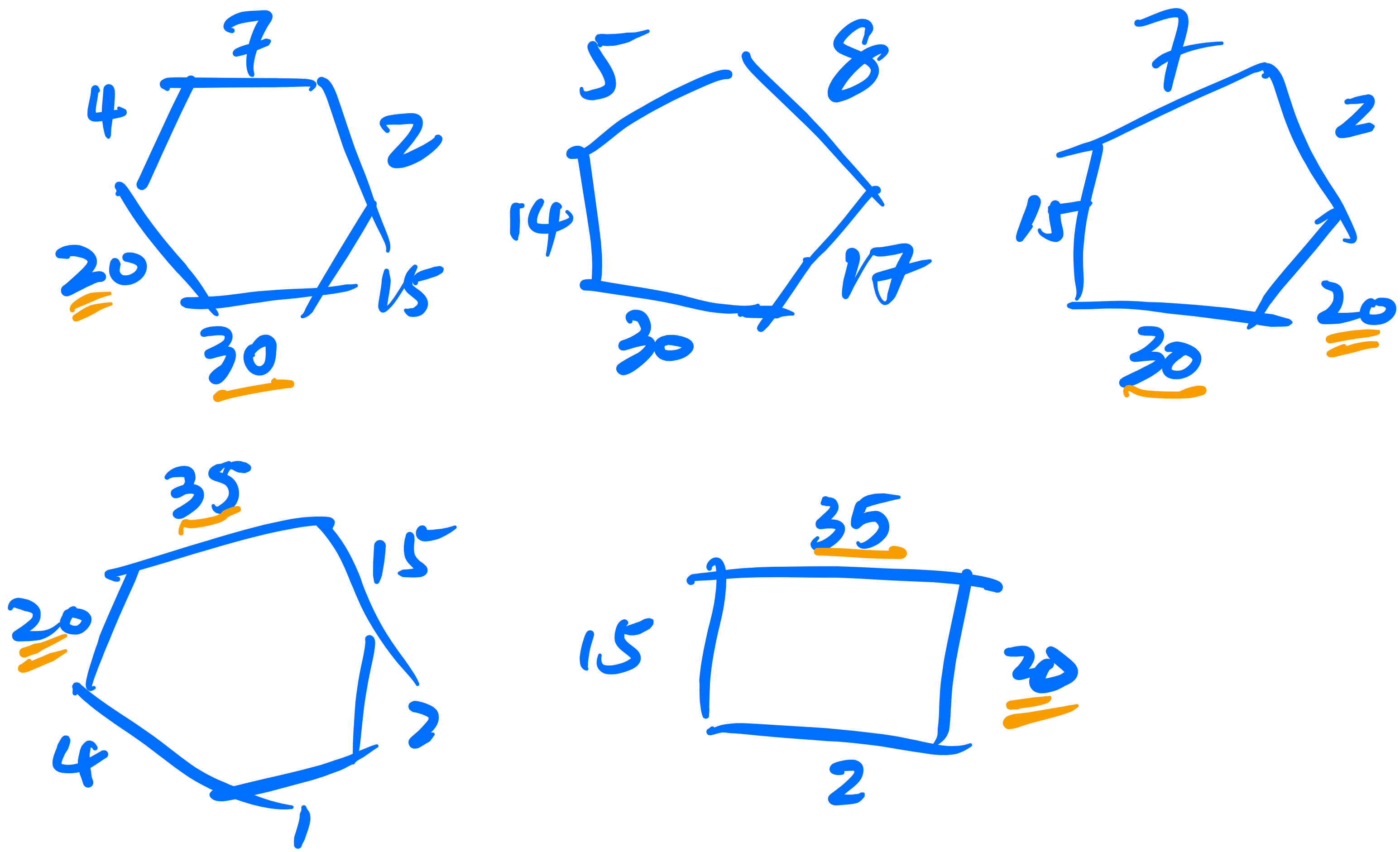
cubical polyhedra

(cubical) facets

"colors" $\in$

$[n] = \{1, \cdots, n\}$

• Some issues:



↳ "supercoloring"

- Why not using Haglund–Wise's criteria?

  "hyperplane subgroups and double cosets of their pairs are separable $\Rightarrow$ VS"

Not obvious how to construct the desired finite quotients.

Agol's "weak separation" Suppose wall stabilizers are VS

For any $R>0$

$G \curvearrowright X$ as above. Then:

$\exists\, G \xrightarrow{\phi} \mathcal{G}$ word-hyp.

$\mathcal{X} = X / \mathrm{Ker}(\phi)$ $(\mathcal{G} \curvearrowright \mathcal{X})$.

such that every R-nbd of any wall in $\mathcal{X}$ is embedded (no self-intersection) and compact.

$\mathcal{G} \curvearrowright \mathcal{X}$

$P(\mathcal{X}) = \{\text{cub. polyh. in } \mathcal{X}\}$

$\Gamma = \Gamma(\mathcal{X}) : V(\Gamma) = \{\text{walls of } \mathcal{X}\}$

$E(\Gamma) = \{\text{wall pairs of dist} \leq R\}$

$$C_n(\Gamma) = \left\{ \begin{array}{l} c : V(\Gamma) \longrightarrow \{1, \dots, n\} \\ \text{s.t. } c(v) \neq c(w) \text{ for all } \{v, w\} \in E(\Gamma) \end{array} \right\}$$

On $V(\Gamma) \times Cn(\Gamma)$, $(v,c) \simeq (v,d) :\Leftrightarrow$

- $c(v) = d(v) = 1$; OR
- $c(v) = d(v) > 1$ AND for any $\{v,w\} \in E(\Gamma)$ where $c(w) < c(v)$ or $d(w) < d(v)$, $c(w) = d(w)$ holds.

On $P(\mathcal{H}) \times Cn(\Gamma)$, $(P,c) \simeq (P,d) :\Leftrightarrow$

for any facet F, $(F,c) \simeq (F,d)$ holds. (i.e. $(\mathrm{wall}(F), c) \simeq (\mathrm{wall}(F), d)$)

resp. $V(\Gamma) \times Cn(\Gamma)$

Note $G \curvearrowright P(\mathcal{H}) \times Cn(\Gamma)$: $\alpha.(P,c) = (\alpha(P), c \circ \alpha^{-1})$

Orbits of $\simeq$-equv. classes form wider classes defining a wider equv. relation $\sim$

$(P,c) \sim (P',c')$, resp. $(F,c) \sim (F',c')$.

Any $\sim$ - eqv. class $[P, c]$ is thought of as a "supercolored cubical polyhedron"

A collection of sup. col. cub. polyh is hence given as

$$\omega : \mathcal{P}(\mathcal{X}) \times Cn(\Gamma)/_\sim \longrightarrow \mathbb{Z}_{\geq 0}$$

compatibility condition

for any (F, c) and P, Q sharing F,

$$\sum_{[P,d]:\,(F,d)\sim(F,c)} \omega([P,d]) = \sum_{[Q,d]:\,(F,d)\sim(F,c)} \omega([Q,d])$$

Agol: ω exists.

Overall structure of the proof:

Induction on $\dim X$:

<u>Step n</u>: $R(G, X)$.

$$\begin{gathered} R(G,X) \\ \| \\ R \gg 0 \end{gathered}$$

- Choose some $R \gg 0$
- $\mathcal{G} \curvearrowright \mathcal{X}$. (walls are VS by induction) and Wise's thm.
- Γ. $\mathcal{P}$. $\sim$. $C_N(\Gamma)$.

$N = \mathrm{maxval}(\Gamma) + 1$.

- Obtain ω.
- $\tilde{\mathcal{V}}_N$ = disj. union as given by ω.

glue

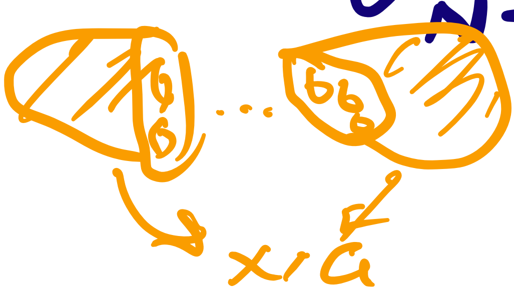
and virtually glue using SQT

cov .... cov

to obtain

$\tilde{\mathcal{V}}_{N-1}, \tilde{\mathcal{V}}_{N-2}, \dots, \tilde{\mathcal{V}}_0$

$\tilde{\mathcal{V}}_0 = \tilde{\mathcal{V}}_{[\text{final}]}$



- <u>Claim</u> : $\mathcal{U}$ is special.

Done.

Thank you!