Subgroup separability and profinite completion of gps

1. subgroup separability (classical)
2. profinite completions of gps (fancier)
3. 1 and 2 on 3-mfld gps : Grothendieck rigidity

All groups are finitely generated and residually finite.

1. Subgroup separability of gps

Peter Scott 1978
Subgps of surface groups are almost geometric

§1 Algebraic definition.

Def. A group G is residually finite if
for $\forall g \in G \setminus \{e\}$ $\exists$ a homomorphism $f: G \to Q$ to a finite gp s.t. $f(g) \neq e$

($\overset{\text{exercise.}}{\Longleftrightarrow}$ $\exists$ $G' < G$ finite index s.t $g \notin G'$)

Example: 1) $\mathbb{Z}^n$ $\forall g \in \mathbb{Z}^n \setminus \{0\}$ for large M, $g \notin M\mathbb{Z}^n$

2) Finitely generated linear groups $G < GL_n(\mathbb{C})$
If $G < GL_n(\mathbb{Z})$, consider $G \to GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/M\mathbb{Z})$, $g \neq e$

General case: use some field theory

for now

3) Free groups, surface groups
(fundamental groups of surfaces)

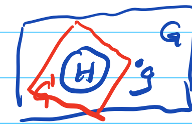
4) 3-manifold gps (Hempel 1984 + geometrization 2002 perelman.)

Nonexample: infinite simple groups

Conj: All hyp groups are residually finite.

Def A group G and a subgroup $H<G$, H is separable in G if

$\forall g \in G \setminus H$, $\exists$ a homo. $f: G \to Q$ to a finite gp s.t. $f(g) \notin f(H)$

($\Leftrightarrow$ $\forall g \in G\setminus H$, $\exists$ $G' < G$ finite index s.t. $g \notin G'$, $H \subset G'$ )

A group G is LERF (locally extended residually finite) if all finite generated subgps of G are separable in G

- G is residually finite $\Leftrightarrow$ $\{e\} < G$ is separable in G

Any finite index $H<G$ is separable.

§2 Topological meaning

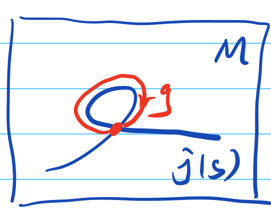
All above definitions are pure group theory, where is topology?

Rough idea: $j: \overset{\text{compact}}{S} \looparrowright M$ π_1-inj immersion $\left(\text{e.g. } \begin{array}{l} S^1 \looparrowright M^2 \\ \Sigma^2 \looparrowright M^3 \end{array}\right)$

If $j_*(\pi_1(S)) < \pi_1(M)$ is separable, then

then $\exists$ a finite cover $M' \to M$ s.t. j lifts to an embedding into M', up to homotopy

$$\begin{tikzcd} & M' \arrow[d, "\text{finite cover}"] \\ S \arrow[ur, "\text{embedding}"] \arrow[r, "j"'] & M \end{tikzcd}$$



$g \in \pi_1(M) \setminus j_*(\pi_1(S))$

separability $\Rightarrow$ $M_1 \to M$ finite cover

s.t. $j_*(\pi_1(S)) < \pi_1(M_1)$

$g \notin \pi_1(M_1)$

$$\begin{tikzcd} & M_1 \arrow[d] \\ S \arrow[ur] \arrow[r, "j"'] & M \end{tikzcd}$$

kill one self-intersection

Keep doing it.

Precise description

Lemma (Scott 1978) X is a nice top. space (e.g. CW-complex)

$H < \pi_1(X)$ subgroup $\quad ([\pi_1(X):H]=\infty)$

$\pi: X_H \to X$ covering space corresponding to H

then the following are equiv.

1) H is separable in $\pi_1(X)$
2) For any $K \subset X_H$ compact, $\exists$ an intermediate finite cover X' of $\pi: X_H \to X$ s.t. $P|_K : K \to X'$ is an embedding

$$\begin{tikzcd} K \subset X_H \arrow[d] \arrow[dd, bend left, "\pi"] \\ X' \arrow[d, "\text{finite cover}"'] \\ X \end{tikzcd}$$

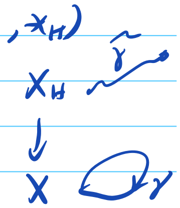
compact

Pf: Give X_H and X basepoints $*_H$ and $*$ s.t. $\pi(*_H) = *$

2) $\Rightarrow$ 1)

For any $g \in \pi_1(X) \setminus H$, take a path $\gamma: (I,\{0,1\}) \to (X,*)$ represents g, lift to $\tilde{\gamma}: (I,\{0\}) \to (X_H, *_H)$

$g \notin \pi_1(X_H) = H \Rightarrow \tilde{\gamma}(0) \neq \tilde{\gamma}(1)$



Apply 2) to $K = \{\tilde{\gamma}(0), \tilde{\gamma}(1)\} \subset X_H$ compact.

$\exists$ X' intermediate finite cover of $X_H \to X$

$P(\tilde{\gamma}(0)) \neq P(\tilde{\gamma}(1))$, for $P \circ \tilde{\gamma}: 0, 1$

$g \notin \pi_1(X') \Leftarrow$ have different image

$$\begin{tikzcd} X_H \arrow[dr, "P"] \arrow[dd, "\pi"'] & \\ & X' \arrow[dl] \\ X & \end{tikzcd}$$

1) $\Rightarrow$ 2) Take $\hat{X} \xrightarrow{P} X$ universal cover

$$\begin{tikzcd} \hat{X} \arrow[rr, "P"] \arrow[dr, "q"'] & & X \\ & X_H \arrow[ur, "\pi"] \arrow[r] & X' \arrow[u] \end{tikzcd}$$

$\exists K' \subset \hat{X}$ compact s.t. $q(K') \supset K$

consider $\pi_1(X)$ as deck-trans gp of $\pi: \hat{X} \to X$

K' compact $\Rightarrow A = \{g \in \pi_1(X) \mid g(K') \cap K' \neq \emptyset\}$ is finite

$\Rightarrow A \setminus H$ finite

separability $\Rightarrow \exists X' \to X$ intermediate finite cover of $X_H \to X$

s.t. ① $\pi_1(X') \supset H$ ② $A \setminus H \cap \pi_1(X') = \emptyset$

suppose $K \to X'$ is not injective.

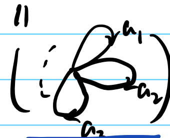
$$\begin{array}{ccccl}
\tilde{X} & \supset K' \ni & y_1, & y_2 & \quad y_1 = g y_2 \text{ for some } g \in \pi_1(X') \\
\downarrow & & \downarrow & \downarrow & \\
X_H & \supset K \ni & x_1 \neq & x_2 & \quad \Rightarrow g \notin H,\ g \in A \\
\downarrow & & \downarrow & \downarrow & \quad g \in A \setminus H \quad g \in \pi_1(X') \\
X' & & \text{if same image} & & \quad \text{contradict with ② impossible!} \\
\downarrow & & & & \\
X & & & &
\end{array}$$

§3 Examples

Free groups and surface groups are LERF ($\Rightarrow$ residually finite)

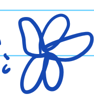
Thm 1. Free groups are LERF

proof: Free gp: F_n, $H < F_n$ finitely generated subgp

$F_n = \pi_1($  $) = \pi_1(X)$

$X_H \longrightarrow X$ covering space corresponding to $H < F_n$

X_H ↑ a graph

- H finitely generated $\Rightarrow$ $\exists f:$  $\longrightarrow X_H$ π_1-surj

  compact $\mathrm{im}(f)$ compact

- For any $K \subset X_H$ compact, enlarge K

  s.t ① $\mathrm{im}(f) \subset K$

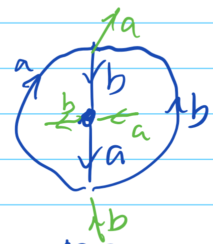
  ② $\boxed{K \text{ is a finite subgraph of } X_H}$ geometrically nice.

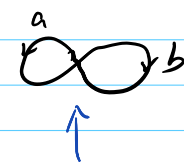
Add edges to K to get a finite cover X' of X

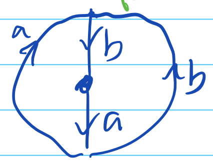
① $\Rightarrow \pi_1(X') > H \Rightarrow X'$ is an intermediate finite cover of $X_H \to X$

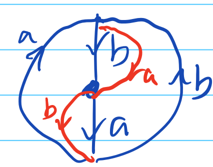
$X' = K + \text{edges} \Rightarrow K$ embeds into X' ∎

Example $\langle a^2b,\ ab^2a^2b\rangle < F_2 = \langle a,b \mid \cdot \rangle$

$X_H =$ 

$X =$ 

$K =$ 

$X' =$ 

Thm 2 Surface groups are LERF (Scott 1978)

$\Rightarrow 1 \to \mathbb{Z} \to G \to H \to 1$ H LERF $\Rightarrow$ G LERF

Agol (2012): hyperbolic 3-mfld gps are LERF

$1 \to H \to G \to \mathbb{Z} \to 1$ H LERF $\not\Rightarrow$ G LERF

$F_2 \times F_2$ is not LERF