2. profinite completions of groups

All groups are finitely generated and residually finite

§1. profinite completions of groups

$\boxed{\text{Ribes - Zalesskii profinite groups}}$

- A group $G \rightsquigarrow \mathcal{C}(G) = \{ G/N \mid N \trianglelefteq G \text{ finite index} \}$

collection of iso. classes of finite quotients of G — "genus" of G

Question: What properties of G are determined by $\mathcal{C}(G)$?

We do need residually finiteness:

$\mathcal{C}(\text{infinite simple gp}) = \{\text{trivial gp}\} = \mathcal{C}(\text{trivial gp})$

Exercise: if G, H Residually finite, $\mathcal{C}(G) = \mathcal{C}(H) \Rightarrow |G| = |H|$

$\mathcal{C}(G)$ has very little structure, but it is better organized by the:

profinite completion of G

- $\hat{G} = \varprojlim_{\substack{N \trianglelefteq G \\ \text{finite index}}} G/N$ $\quad N_1, N_2 \overset{f.i.}{\trianglelefteq} G \quad N_1 < N_2$

$\varphi_{N_1 N_2} : G/N_1 \overset{\text{surj.}}{\twoheadrightarrow} G/N_2 \qquad gN_1 \to gN_2$

$$\hat{G} = \varprojlim_{\substack{N \trianglelefteq G \\ f.i.}} G/N \overset{\text{def}}{=} \left\{ (g_N N) \in \prod_{\substack{N \trianglelefteq G \\ f.i.}} G/N \;\middle|\; \begin{array}{l} N_1 < N_2 \quad N_1, N_2 \overset{f.i.}{\trianglelefteq} G \\ \varphi_{N_1 N_2}(g_{N_1} N_1) \\ = g_{N_2} N_2 \end{array} \right\} \subseteq \prod_{\substack{N \trianglelefteq G \\ f.i.}} G/N$$

Example G finite, $\hat{G} \cong G$

$$\hat{\mathbb{Z}} = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z} \cong \prod_{p \text{ prime}} \Big(\underbrace{\varprojlim \mathbb{Z}/p^k\mathbb{Z}}_{\cong \mathbb{Z}_p \text{ p-adic integers.}} \Big)$$

Natural homomorphisms

$i : G \to \hat{G}$, $\quad g \to (gN)$

$\forall N_0 \overset{f.i.}{\trianglelefteq} G \quad \pi_{N_0} : \hat{G} \to G/N_0$, $\quad (g_N N) \to g_{N_0} N_0$

Facts/exercises:

1) $\hat{G}$ is closed $\prod G/N$

$\Rightarrow \hat{G}$ is a compact, Hausdorff, totally disconnected top. group

2) $i : G \to \hat{G}$ is injective $\Leftrightarrow$ G is residually finite.

$i(G) \subset \hat{G}$ is dense.

3) $\forall N_0 \trianglelefteq G$ f.i. $\pi_{N_0} : \hat{G} \to G/N_0$ is surj.

4) $H < G$ is separable $\Leftrightarrow$ H is closed in G, under the subspace topology of $G \subseteq \hat{G}$

$i: G \to \hat{G}$

1-1 correspondence of finite index subgps $\underset{\text{f.g.}}{G}$ vs $\hat{G}$

$H \overset{\text{f.i.}}{<} G \longrightarrow \overset{\text{closure in } \hat{G}}{\overline{i(H)}} < \hat{G}$ f.i. open

$K \underset{\text{open}}{\overset{\text{f.i.}}{\leq}} \hat{G} \longrightarrow K \cap G \overset{\text{f.i.}}{<} G$

Assume normal, $G/H \cong \hat{G}/\overline{H}$

(Nikolov-Segal 2007 G f.g. any f.i subgp of $\hat{G}$ is open.)

Prop: G_1, G_2 both finitely generated
$\mathcal{C}(G_1) = \mathcal{C}(G_2) \iff \hat{G}_1 \cong \hat{G}_2$

Pf: "$\Leftarrow$" $j: \hat{G}_1 \to \hat{G}_2$ iso
$\forall N_1 \lhd G_1$ finite index

$G_1/N_1 \cong \hat{G}_1/\overline{N}_1 \cong \hat{G}_2/K_2 \cong G_2/G_2 \cap K_2$ (K_2: exist)

($\overline{N}_1$: f.i. open normal subgp of $\hat{G}_1$) $\mathcal{C}(G_1) \subseteq \mathcal{C}(G_2)$

"$\Rightarrow$" G finitely generated $\Rightarrow \forall n \in \mathbb{N}$, G has finitely many normal subgps of index $\leq n$
(fix finite gp Q, $G \to Q$ finitely many homo)

$\mathcal{C}(G_1) = \mathcal{C}(G_2)$
$\Rightarrow \forall n \in \mathbb{N}$

$$G_1 \Big/ \bigcap_{\substack{N \lhd G_1 \\ [G_1:N] \leq n}} N \cong G_2 \Big/ \bigcap_{\substack{N \lhd G_2 \\ [G_2:N] \leq n}} N$$

max quotient of G_1 G_2 s.t. $\cap$ index $\leq n$ normal subgps is trivial

$= G_1/K_{1,n}$ $\qquad = G_2/K_{2,n}$

$\Rightarrow \underset{\cong \hat{G}_1}{\varprojlim G_1/K_{1,n}} \cong \underset{\cong \hat{G}_2}{\varprojlim G_2/K_{2,n}}$ $\qquad \forall N \overset{\text{f.i.}}{\lhd} G_1 \ \exists n$ s.t. $N > K_{1,n}$ $\square$

Natural definition: G f.g. residually finite gp. G is profinite rigid if for any H f.g. residually finite gp. $\hat{G} \cong \hat{H}$ $(\mathcal{C}(G) = \mathcal{C}(H)) \Rightarrow G \cong H$

Different category: f.g. residually finite gps, 3-mfld gps

Fact: G f.g. abelian gp, then G is profinite rigid
- If $\mathcal{C}(G) = \mathcal{C}(H)$
  H is abelian: if not, $\exists h_1, h_2$ $[h_1, h_2] \neq e$
  residually finite: $\exists N \overset{\text{f.i.}}{\lhd} H$ $[h_1,h_2]N \neq e \in H/N$
  $\Rightarrow H/N$ is not abelian $\Rightarrow \mathcal{C}(G) \neq \mathcal{C}(H)$
  Classification of f.g. abelian gp $\Rightarrow G \cong H$

consequence: $\widehat{G}$ determines G^{ab}, so $\widehat{F_m} \cong \widehat{F_n} \Rightarrow F_m \cong F_n$

$\widehat{\pi_1(\Sigma_g)} \cong \widehat{\pi_1(\Sigma_h)} \Rightarrow \pi_1(\Sigma_g) \cong \pi_1(\Sigma_h)$

Counterexample $\exists\ G_1, G_2$ s.t.   virtually cyclic $1 \to \mathbb{Z}/11\mathbb{Z} \to G_1 \to \mathbb{Z} \to 1$, $\to G_2 \to$

s.t. $\widehat{G_1} \cong \widehat{G_2}$ but $G_1 \not\cong G_2$

$\exists$ pairs of Seifert mfld, solmfld, graph mfld gps

s.t. $\widehat{G_1} \cong \widehat{G_2}$ but $G_1 \not\cong G_2$ (Gareth Wilkes)

Open cases for profinite rigidity:

free groups, surface groups, hyp 3-mfld groups

§2. Induced homomorphisms on profinite completions

$u: H \to G$ a homo. induces $\hat{u}: \hat{H} \to \hat{G} = \varprojlim G/N$

$(h_K K) \to (g_N N)$

$\forall N \lhd G$ f.i., $u^{-1}(N) \lhd H$ f.i.

$\hat{H} \to H/u^{-1}(N) \xrightarrow{u \text{ induces}} G/N$ } compatible with $G/N_1 \to G/N_2$

Get $\hat{u}: \hat{H} \to \hat{G}$.

Grothendieck (1970) G, H both finitely presented residually finite

$u: H \to G$ a homo s.t. $\hat{u}: \hat{H} \to \hat{G}$ is an iso, is u an iso?

Can assume u is inj,

$$\begin{tikzcd}
h \in \ker u \quad H \arrow[r, "u"] \arrow[d, hook, "i_H"] & G \arrow[d] \arrow[dr, bend left] & \\
\hat{H} \arrow[r, "\hat{u}"] & \hat{G} & e
\end{tikzcd}$$

$\underset{\substack{\| \\ e}}{i_H(h)} \in \ker \hat{u}$

can assume $i: H \to G$ inclusion

Def. A f.g. residually finite gp G is <u>Grothendieck rigid</u> if for any f.g. subgp $H \lneq G$, $\hat{i}: \hat{H} \to \hat{G}$ is not an iso

Thm (Platonov – Tavgen 1990 / Bridson – Grunewald 2004)

Not all finitely generated / finitely presented gps are G. rigid

$F_2 \times F_2$

Lemma (Long–Reid 2011)

If $H < G$ separable and $H \neq G$, $\hat{i} : \hat{H} \to \hat{G}$ is not an iso.

Pf $H \neq G \Rightarrow$ $\exists g \in G \setminus H$

H separable in $G \Rightarrow$ $\exists N \trianglelefteq G$ f.i. s.t. $gN \notin HN$ in G/N

Consider

$$\begin{tikzcd} & \hat{H} \arrow[r, "\hat{i}"] \arrow[d, "surj"] \arrow[drr, bend left, "\text{if } \hat{i} \text{ is iso, is surj.}"] & \hat{G} \arrow[d, "surj"] \\ H \arrow[r] \arrow[rr, bend right, "\text{is not surj}"'] & H/N \cap H \arrow[r] & G/N \end{tikzcd}$$

contradiction! $\square$

Coro: Free gps, surface gps, hyp 3-mfld gps are G. rigid

(since all are LERF)

To prove G is G. rigid, only need to consider nonseparable subgps

Hopeless for general groups, but possible for 3-mfld gps.

Thm (S. 2021) All (f.g.) 3-mfld gps are G. rigid.