3. Grothendieck rigidity of 3-mfld gps

Thm (S. 2021) All (f.g.) 3-mfld gps are Grothendieck rigid.

- Goal: G: 3-mfld gp, $H < G$ f.g. subgp $H \neq G$ then $\hat{i}: \hat{H} \to \hat{G}$ is not an iso.
- Only need to consider $\boxed{H < G \text{ nonseparable.}}$ $\Rightarrow [G:H] = \infty$

§1 Basic 3-mfld topology

nice behavior on π_1

M connected 3-mfld with finitely generated π_1

$\downarrow$ orientable double cover / index-2 subgp

M also orientable

$\downarrow$ Scott 1973 $\exists$ compact codim-0 submfld $N \subset M$ s.t. $\pi_1(N) \to \pi_1(M)$ is an iso

M also compact (possibly some ∂-component has $g \geq 2$)

$\downarrow$ sphere/disc decomposition: free product

M irreducible, ∂-incompressible or $\cancel{S^2 \times S^1}$

(Any embedded S^2 bound a 3-ball, each ∂-comp is π_1-inj)

$\pi_2 = 0$ if $|\pi_1| = \infty \Rightarrow$ aspherical $\pi_n = 0$ $n \geq 2$

paste groups along $\mathbb{Z}^2$ graph-of-group structure

torus decomp. embedded π_1-inj torus

M is either Seifert (circle bundle over 2-orbifold/surface.)

or atoroidal (any π_1-inj $T^2 \hookrightarrow M$ is homotopic to ∂M)

$\Rightarrow$ hyperbolic possible vol $= \infty$

behavior is bad.

§2 LERFness of 3-mfld gps

All 3-mflds are compact orientable, connected

LERF gps

(Scott 1978) Seifert mflds have LERF gps

(Agol 2012) Hyp 3-mflds have LERF gps

non LERF gps

M^3: $\partial = \emptyset$ or $\sqcup T^2$, each piece of torus decomp is Seifert.

torus decomp $\neq \emptyset$, not a Sf mfld

(Burns-Karrass-Solitar 1987 — some; Rubinstein-Wang 1997 — some; Niblo-Wise 2001 — all) graph mfld gps are not LERF

$\to$ nonseparable subgps are surface gps

M^3: $\partial = \phi$ or $\amalg T^2$, torus decomp $\neq \phi$
has a hyp piece.

(Liu 2017) some
(Sun 2019) all } mixed 3-mfld gps are not LERF

(☆) irreducible, ∂-incompressible
has nontrivial torus decomp., is not a Sol mfld

(Sun 2020) A 3-mfld satisfies ☆ has LERF $\pi_1 \Leftrightarrow$
For any two adjacent pieces share a torus, at least one has a ∂-comp. with $g \geq 2$

§3 Graph of groups and its profinite analogy

A graph-of-group $(T, \mathcal{G})$

A graph T, each edge is oriented

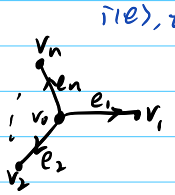
each vertex v of T $\rightsquigarrow$ vertex group $\mathcal{G}_v$

each edge e of T $\rightsquigarrow$ edge group $\mathcal{G}_e$

$\mathcal{G}_e \xrightarrow{\text{injective homomorphisms}} \mathcal{G}_{i(e)}$, $\mathcal{G}_e \rightarrow \mathcal{G}_{t(e)}$

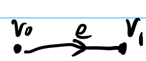
$i(e), t(e)$: initial, terminal vertices of e.

Example:

$T =$ 

$\mathcal{G}_{v_0} = \{e\}$ $\mathcal{G}_{v_i} = G_i$

$\mathcal{G}_{e_i} = \{e\} \rightarrow \mathcal{G}_{v_0}$, $\mathcal{G}_{e_i} \rightarrow \mathcal{G}_{v_i}$ trivial homomorphisms

$T =$ 

$T =$ 

$\mathcal{G}_e \rightrightarrows \mathcal{G}_v$
two inj homo, usually distinct.

$\mathcal{G}_{v_0} = F_2 = \langle a, b \rangle$

$\mathcal{G}_{v_1} = F_2 = \langle c, d \rangle$

$\mathcal{G}_e = \mathbb{Z} \rightarrow \mathcal{G}_{v_0} = \langle a, b \rangle$, $\mathcal{G}_e = \mathbb{Z} \rightarrow \mathcal{G}_{v_1} = \langle c, d \rangle$

$[a,b]$, $[c,d]$

Realize as a graph-of-space

Each vertex $v \longrightarrow K(\mathcal{G}_v, 1)$

Each edge $e \longrightarrow K(\mathcal{G}_e, 1) \times I$

$K(\mathcal{G}_e, 1) \times 0 \longrightarrow K(\mathcal{G}_{i(e)}, 1)$

$K(\mathcal{G}_e, 1) \times 1 \longrightarrow K(\mathcal{G}_{t(e)}, 1)$

realize inj homo.

paste together to get a space K

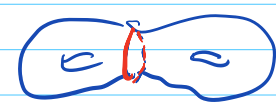
$$K = \Big(\coprod_{\text{vertex}} K(\mathcal{G}_v, 1)\Big) \coprod_{\text{pasting map}} \Big(\coprod_{\text{edge } e} K(\mathcal{G}_e, 1) \times I\Big)$$

K is aspherical.

Graph product of $(\Gamma, \mathcal{G})$

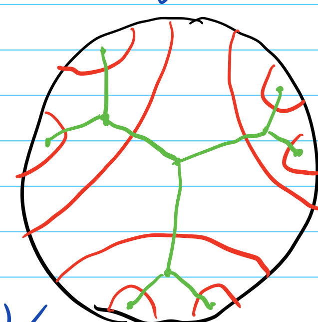
$\pi_1(\Gamma, \mathcal{G}) \overset{\text{def}}{=} \pi_1(K)$ $\quad$ $G_v, G_e \overset{\text{inj}}{\hookrightarrow} \pi_1(\Gamma, \mathcal{G})$

Above examples: $\bigvee_{i=1}^{n} K(G_i, 1)$,



Universal cover $\tilde{K}$ of K:

has an induced graph-of-space structure, with dual graph a tree T



$\pi_1(K)$ acts on T

vertex stabilizers: conjugates of $\mathcal{G}_v$

edge stabilizers: conjugates of $\mathcal{G}_e$.

abstract view of T

vertices: $\coprod_{\text{vertex } v} \pi_1(\Gamma, \mathcal{G}) / \mathcal{G}_v$ $\quad$ edges: $\coprod_{\text{edges } e} \pi_1(\Gamma, \mathcal{G}) / \mathcal{G}_e$

irreducible, ∂-incompressible 3-mfld:

has a graph of space structure, given by torus decomp.

$\pi_1(M)$: π_1 of a graph-of-group, edges groups are $\mathbb{Z}^2$

• profinite version. start with same $(\Gamma, \mathcal{G})$

same graph Γ, vertex gp $\hat{\mathcal{G}}_v$

edge gp $\hat{\mathcal{G}}_e$ $\quad$ $\hat{\mathcal{G}}_e \to \hat{\mathcal{G}}_{i(e)}$, $\hat{\mathcal{G}}_e \to \hat{\mathcal{G}}_{t(e)}$ $\quad$ (may not be inj)

inj under nice conditions.

Algebraic construction $\underset{\text{profinite gp}}{\pi_1(\Gamma, \hat{\mathcal{G}})} \xleftarrow[\text{be inj.}]{\text{may not}} \hat{G}_v, \hat{G}_e$

Under nice conditions

$\widehat{\pi_1(\Gamma, \mathcal{G})} \cong \pi_1(\Gamma, \hat{\mathcal{G}})$ acts on a profinite tree $\hat{T}$

vertices: $\quad$ edges:

$\coprod_{\text{vertex } v} \pi_1(\Gamma, \hat{\mathcal{G}}) / \hat{\mathcal{G}}_v$ $\quad$ $\coprod_{\text{edge } e} \pi_1(\Gamma, \hat{\mathcal{G}}) / \hat{\mathcal{G}}_e$

Nice conditions

1) $\pi_1(\mathcal{P}, \mathcal{G})$ is residually finite

2) Any $m \in \mathcal{P}$ (a vertex or an edge), any finite index $K < G_m$
   K is separable in $\pi_1(\mathcal{P}, \mathcal{G})$

irreducible ∂-incomp. 3-mfld M with torus decomposition satisfies these nice conditions

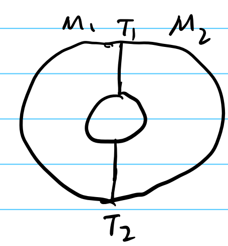
§4. How does a nonseparable gp look like?

They come from virtual fiber surfaces in pieces of M! (Rubinstein-Wang 1997, Liu 2017)

skip.

Example $M = M_1 \cup M_2$

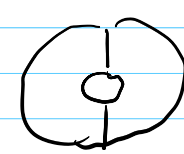
$M_1 \cap M_2 = T_1 \cup T_2$ two tori



$j: \Sigma \looparrowright M$ properly immersed, π_1-inj, $a, b \in \mathbb{Z}_{>0}$, $a \neq b$

$\Sigma = (\Sigma_{1,1} \cup \Sigma_{1,2}) \cup (a+b \text{ copies of } \Sigma_2)$

$\Sigma_{1,1}, \Sigma_{1,2}$:



Σ_2 :

$c_1 \subset T_1$, $c_2 \subset T_2$ essential s.c.c.

$\Sigma_{1,1} \cap T_1$: $\qquad$ $\Sigma_{1,2} \cap T_1$:

$\Sigma_{1,1} \cap T_2$: $\qquad$ $\Sigma_{1,2} \cap T_2$:

$\Sigma_2 \cap T_1$ $\qquad$ $\Sigma_2 \cap T_2$

Then $j_*(\pi_1(\Sigma)) < \pi_1(M)$ is not separable

Idea of proof :

suppose separable,

. lift Σ

. take cyclic cover

. get a contradiction by computing covering degrees of

∘ $H < \pi_1(M)$ nonseparable, finitely generated

$\pi_1(M)$ has a graph-of-group structure

$M_H \to M$ covering space corresponding to $H < \pi_1(M)$

M_H: graph-of-space structure

$\rightsquigarrow$ H has a graph-of-group structure, with edge groups: $\{e\}, \mathbb{Z}, \mathbb{Z}^2$

H acts on a tree T

(Sun 2020) For M satisfies ✱, $H < \pi_1(M)$ f.g., there is a characterization of the separability of H

§5. Grothendieck rigidity of 3-mfld gps

Consequence: If $H < \pi_1(M)$ nonseparable, there exists a vertex stabilizer $H^v < H$ that is free non-abelian, s.t.

1) either H^v fixes a vertex $v \in T$ that projects to a hyperbolic piece $M^v \subset M$ s.t. $H^v \subset \pi_1(M^v)$ is a virtual fiber subgroup

2) or H^v fixes $v \in T$ that projects to a Seifert piece $M^v \subset M$ s.t. $H^v \subset \pi_1(M^v) \to \pi_1(\underset{\text{base orbifold}}{O^v})$ is inj. $\left(\begin{array}{l} H^v \cap \text{fiber subgp} \\ = \{e\} \end{array} \right)$

key: $\boxed{N_H(H^v) = H^v, \quad [N_G(H^v) : H^v] = \infty}$ $\quad G = \pi_1(M)$

normalizer. consider H action on T, edge groups are abelian. $\quad$ by above description.

These normalizer behaviors pass to profinite world. $G = \pi_1(M)$

suppose $\hat{i}: \hat{H} \to \hat{G}$ is an iso,

1) Good separability properties:
$$\begin{tikzcd} & \widehat{H^v} \arrow[dl] \arrow[dr] & \\ \hat{H} \arrow[rr, "\hat{i}"] & & \hat{G} \end{tikzcd}$$
both injective.

2) $\hat{H}$ acts on a profinite tree $\hat{T}$ (H satisfies the "nice" properties)

$\widehat{H^v}$ is a vertex stabilizer of $\hat{v} \in \hat{T}$

3) edge groups of $\hat{H}$-action on $\hat{T}$ are abelian

$N_{\hat{H}}(\widehat{H^v}) = \widehat{H^v}$

4) H^v is separable in G $\Rightarrow$ $[N_{\hat{G}}(\widehat{H^v}) : \widehat{H^v}] = \infty$ $\Big\}$ $\Rightarrow$ $\hat{i}: \hat{H} \to \hat{G}$ is not an iso. □