## (II) Positivity of direct images (cf. Hacon-Popa-Schnell)

**Thm**: Let $\pi: X \to Y$ be a proper fibration between two Kähler mfds.

$L \to X$ holo line bundle, $i\Theta_{h_L}(L) \geq 0$ ($h_L$ sing)

$e_L$ holo base of $L$, $|e_L|^2_{h_L} = e^{-\varphi}$ $\Rightarrow$ $\varphi$ psh

$\mathcal{I}(\varphi)_x := \{ f \in \mathcal{O}_{X,x} \mid \int_{U_x} |f|^2 e^{-\varphi} < +\infty \}$.

If: $\pi_*(K_{X/Y} \otimes L \otimes \mathcal{I}(\varphi)) \neq 0$

$\Leftrightarrow$ $X_y$ generic fiber

$H^0(X_y, K_{X/Y} \otimes L \otimes \mathcal{I}(\varphi|_{X_y})) \neq 0$.

Then: • $K_{X/Y} \otimes L$ psef

• $(\pi_*(K_{X/Y} \otimes L \otimes \mathcal{I}(\varphi)), h) \succeq 0$ Griffith.

In general, $m \in \mathbb{N}^*$. $\pi_*(mK_{X/Y} \otimes L \otimes \mathcal{I}_m(\varphi)) \neq 0$

$\Rightarrow$ • $mK_{X/Y} + L$ psef

• $(\pi_*(mK_{X/Y} \otimes L \otimes \mathcal{I}_m(\varphi)), h) \succeq 0$.

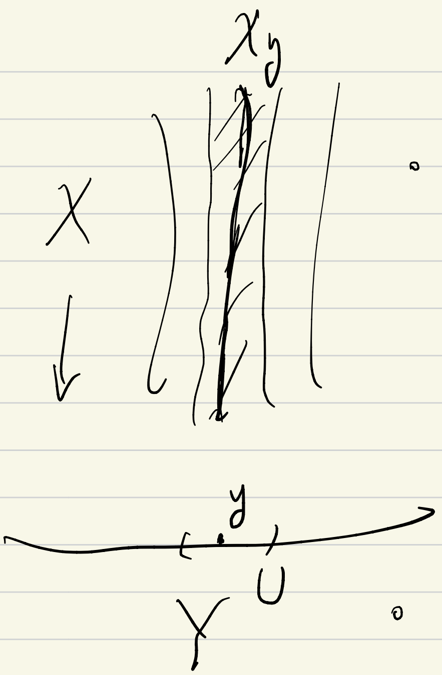
**(I) smooth case** $\pi: X \to Y$ be a submersion, proper

$L \to X$ $i\Theta_{h_L}(L) \geq 0$ $h_L$ $C^\infty$ herm.

**Lemma 1**: • $\pi_*(K_{X/Y} + L)$ is locall free on $Y$.

• $\forall y \in Y$, $(\pi_*(K_{X/Y} + L))_y = H^0(X_y, K_{X/Y} + L)$.

**proof**: $\forall y \in Y$, let $U$ be a topo small nb of $y \in Y$.




- $\forall\ \boxed{s \in H^0(X_y,\ K_{X/Y}+L)}$

  By O-T $\Rightarrow \exists\ \tilde{s} \in H^0(\pi^{-1}(U),\ K_{X/Y}+L)$

  s.t. $\tilde{s}|_{X_y} = s$.

- In particular, let $\boxed{\{s_1, \cdots, s_r\}}$ be a base of $H^0(X_y,\ K_{X/Y}+L)$.

Then O-T: $\exists\ \boxed{\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_r} \in H^0(\pi^{-1}(U), K_{X/Y}+L)$

and $\tilde{s}_i|_{X_y} = s_i$

$\Rightarrow$ For $\underset{\in U}{y'}$ closed to $y$ $\quad \{\tilde{s}_1|_{X_{y'}}, \cdots, \tilde{s}_r|_{X_{y'}}\}$

is linear indep't of $H^0(X_{y'}, K_{X/Y}+L)$

$$\Rightarrow \begin{aligned} Y &\longrightarrow \mathbb{N} \\ y &\longrightarrow \dim H^0(X_y,\ K_{X/Y}+L) \quad \text{l.s.c} \end{aligned}$$

$\Rightarrow$ $y \longrightarrow \dim H^0(X_y,\ K_{X/Y}+L)$ is const.

$\Rightarrow$ $\pi_*(K_{X/Y}+L)$ is locally free, $\quad (\pi_*(K_{X/Y}+L))_y = H^0(X_y,\ K_{X/Y}+L)$ □

$L^2$-metric

**Cor**: ($C^\infty$ $L^2$-metric) We can find a $C^\infty$ herm $h$ on $\pi_*(K_{X/Y}+L)$ as follows:

Lemma

$\forall y \in Y$, $s \in (\pi_*(K_{X/Y}+L))_y = H^0(X_y, K_{X/Y}+L)$

$= H^0(X_y, K_{X_y}+L)$

canonical

$$|s|_h^2 := \int_{X_y} |s|_{h_L}^2$$

$$\begin{array}{c} X^m \\ \downarrow \pi \\ Y^n \end{array}$$

Rk: $s = f \cdot dz \otimes e_L$, with $dz \leftarrow K_{X_y}$, $e_L \leftarrow L$

$|s|_{h_L}^2 := i^{n^2}|f|^2 \cdot |e_L|_{h_L}^2 \cdot dz \wedge d\bar{z}$ ← volume form.

$\Omega_X^{m-n} \otimes \pi^* K_Y \to K_X$

$\alpha \quad \beta \to \alpha \wedge \beta$

$\Rightarrow \varphi: \boxed{\Omega_X^{m-n}/\ker\varphi} \to K_X \otimes (\pi^* K_Y)^*$

$|_{X_y} = K_{X/Y}|_{X_y}$

Exo: $\alpha \in \ker\varphi \iff \alpha|_{X_y} = 0$ as a $(m-n)$-form

$\Omega_X^{m-n}/\ker|_{X_y} = K_{X_y}$

Rk: $h$ is $C^\infty$:

$$\begin{array}{ccc} \pi^{-1}(U) & \overset{C^\infty}{\simeq} & \boxed{\begin{array}{c} X_y \times U \\ \downarrow \\ U \end{array}} \\ \downarrow & & \\ U & & \end{array}$$

Rk: We can reformulate the O-T thm in the following way:

$\overset{0}{\wedge}$ polydisc in $\mathbb{C}^n$, $(z_1, \dots, z_n)$ $dz = dz_1 \wedge \dots \wedge dz_n$.

$\pi: X \longrightarrow \boxed{\Delta^n}$ proper submersion. $X_0 = \pi^{-1}(0)$

$(\pi_*(K_{X/Y} + L), h)$

**O-T** $\Leftrightarrow$ $\forall\ s \in (\pi_*(K_{X/Y} + L))_0 = H^0(X_0, K_{X/Y} + L)$

$\exists\ \hat{S} \in H^0(\Delta^n, \pi_*(K_{X/Y} + L))$

s.t. $i^{n^2} \int_{\Delta^n} \boxed{|\hat{S}|_h^2\, dz \wedge d\bar{z}} \leq C \cdot |S|_h^2$.

$\boxed{\hat{S}(0) = S}$ $\boxed{\hat{S} \wedge dz} \in H^0(X, K_X + L)$

Optimal-OT: $C = i^{n^2} \int_{\Delta^n} dz \wedge d\bar{z}$.

We can take

**Lemma 2**. $V \to X$ holo vector bundle. $\boxed{h\ C^\infty \text{ herm metric on } V}$.

$\boxed{i\Theta_h(V) \geq 0 \text{ Griffith}} \Leftrightarrow \boxed{\begin{array}{l} \forall\ U \subset X \text{ open set.} \\ \forall\ s \overset{\neq 0}{\in} H^0(U, V^*) \\ \boxed{\ln |S|_{h^*}^2} \text{ is psh on } U. \end{array}}$

Rk: $\boxed{\operatorname{rk} V = 1}$

$e_V$ be a base of $V$. $S = \underset{\text{holo}}{f}\, e_V^*$ $e^{\varphi}$

$|e_V|_h = e^{-\varphi}$ $\boxed{i\Theta_h(V) = i\partial\bar{\partial}\varphi} \Leftrightarrow$

$\geq 0$

$\ln |S|_{h^*}^2 = \ln(|f|^2 \cdot |e_V^*|_{h^*}^2)$

$= \ln|f|^2 + \varphi$, psh

Rk: $h$ $C^\infty$ herm $\Rightarrow$ $i\Theta_h(V) \in C^\infty(X, \Lambda_X^{1,1} \otimes \mathrm{End}(V))$

If $h$ is not smooth, $i\Theta_h(V)$ not well defined

$$\bar\partial(h^{-1} \cdot \partial h)$$

But $\ln|s|_{h^*}$ is well defined in general $\square$

Thm: $\pi: X \longrightarrow Y$ proper submersion. $L \to X$ holo line bundle

$i\Theta_{h_L}(L) \geq 0$ $h_L$ $C^\infty$.

Then $(\pi_*(K_{X/Y}+L), h) \geq 0$ Griffith.

Rk: • $L = \mathcal{O}_X$ Griffith.

• Berndtsson: $i\Theta_h(\pi_*(K_{X/Y}+L)) \geq 0$ Nakano

curvature formulate: $i\Theta_h$ can control deformation of $\pi$.

proof: By Lemma 2, it is sufficient to prove

$\forall\, U \subset Y$, open, $\forall\, s \in H^0(U, (\pi_*(K_{X/Y}+L))^*)$

$\ln|s|^2_{h^*}$ is psh on $U$, i.e. $\ln|s|^2_{h^*}$ u.s.c

$\Delta \xrightarrow{p} U$

$p^*\ln|s|^2_{h^*}(0) \leq \frac{1}{\pi}\int p^*$ unit disc

$$\Delta^1 \xrightarrow{p} U$$

$$p^* \ln |S|^2_{h^*}(0) \leq \frac{1}{\pi} \int_{\Delta} p^* \ln |S|^2_{h^*} \, dz \wedge d\bar{z}.$$

$$\begin{array}{ccc} p^*X & & X \\ \downarrow & & \downarrow \pi \\ 0 \in \Delta^1 & \xrightarrow{p} & Y \end{array}$$

$\Leftarrow$ $Y = \Delta^1$

$S \in H^0(\Delta^1, (\pi_*(K_{X/Y} + L))^*)$

$$\boxed{\underline{\ln |S|^2_{h^*}(0)} \leq \frac{1}{\pi} \int_{\Delta^1} \ln |S|^2_{h^*} \, dz \wedge d\bar{z}}.$$

- $E \to X$ $\quad (E, h)$ $\quad (E^*, h^*)$.

$$S \in E^*_x, \quad \underline{|S|^2_{h^*} := \sup_{\substack{\sigma \in E_x \\ |\sigma|_h = 1}} |\langle S, \sigma \rangle|^2}.$$

$E \otimes E^* \to \mathbb{C}$

$\sigma, S \mapsto \underline{\langle \sigma, S \rangle}$

- $\exists \, \boxed{\sigma} \in (\pi_*(K_{X/Y} + L))_0$ s.t. $\underline{|\sigma|_h = 1}$,

$$\boxed{|S|^2_{h^*}(0) = |\langle S, \sigma \rangle|^2}$$

- optimal O-T.: $\tilde{\sigma} \in H^0(\Delta, \pi_*(K_{X/Y} + L))$

$\tilde{\sigma}(0) = \sigma$, $\quad \int_{\Delta^1} |\tilde{\sigma}|^2_h \, dz \wedge d\bar{z} \leq \pi \cdot |\tilde{\sigma}(0)|^2_h$

$$= \pi |\sigma|^2_h = \pi$$

- $S \in H^0(\Delta^1, (\pi_*(K_{X/Y} + L))^*)$, $\quad \tilde{\sigma} \in H^0(\Delta^1, \pi_*(K_{X/Y} + L))$

$$|\langle s, \tilde{\sigma} \rangle| \le |s|_h \cdot |\tilde{\sigma}|_{h^*}$$

$$\langle s, \tilde{\sigma} \rangle \in H^0(\Delta^1, \mathcal{O}_{\Delta^1}).$$

$$\Rightarrow \ln |\langle s, \tilde{\sigma} \rangle|^2 \text{ is psh}.$$

$$\Rightarrow \boxed{\ln |s|^2_{h^*}(0)} = \ln |\langle s, \tilde{\sigma} \rangle|^2 (0) \le \frac{1}{\pi} \int_{\Delta^1} \ln |\langle s, \tilde{\sigma} \rangle|^2 \, dz \wedge d\bar{z}$$

$$\le \frac{1}{\pi} \int_{\Delta^1} \ln \left( |s|^2_{h^*} \cdot |\tilde{\sigma}|^2_h \right) dz \wedge d\bar{z}$$

$$= \frac{1}{\pi} \int_{\Delta^1} \ln |s|^2_{h^*} + \underbrace{\frac{1}{\pi} \int_{\Delta^1} \ln |\tilde{\sigma}|^2_h \, dz \wedge d\bar{z}}_{\overset{\text{Jensen}}{\le} \ln \left( \underbrace{\frac{1}{\pi} \int_{\Delta^1} |\tilde{\sigma}|^2_h \, dz \wedge d\bar{z}}_{\le 1} \right)}$$

Optimal O-T

$$\le \underline{\frac{1}{\pi} \int_{\Delta^1} \ln |s|^2_{h^*} \, dz \wedge d\bar{z}} = 0$$

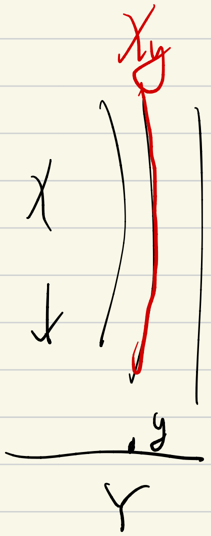
$$\Rightarrow \ln |s|^2_{h^*} \text{ is psh on } Y \quad \square$$

$$\Leftrightarrow \underline{i\Theta_h(\pi_*(\boxed{K_{X/Y}} + L)) \ge 0} \text{ Griffith.} \quad \square$$

**Thm 2**: $\pi: X \to Y$ proper submersion. $i\Theta_{h_L}(L) \geq 0$, $h_L$ $C^\infty$.

If $\pi_*(K_{X/Y}+L) \neq 0$, $\boxed{\exists\, h_B}$ on $K_{X/Y}+L$

s.t. • $i\Theta_{h_B}(K_{X/Y}+L) \geq 0$ ($\Rightarrow K_{X/Y}+L$ psef).




• $\forall\, y \in Y$, $\forall\, s \in H^0(X_y, K_{X_y}+L)$,

$$\boxed{|s|_{h_B} \leq C}$$

i.e., $|s|_{h_B}$ is $L^\infty$ on $X_y$.

Rk: Xiao: In general $|s|_{h_B}$ is not $L^\infty$.

**Def**: (Bergman kernel) $\boxed{X \text{ compact complex mfd}}$.

$L \to X$ holo line bundle, $h_L$ $C^\infty$ herm metric on $L$.

$(\underbrace{H^0(X, K_X+L)}_{\neq 0}, h)$, $h$ ← $L^2$ metric

$s \in H^0(X, K_X+L)$, $\|s\|_h^2 := \int_X \underbrace{|s|^2_{h_L}}_{\text{volume form}}$

Let $\{s_1, \dots, s_r\}$ be an orthonormal base of $(H^0(X, K_X+L), h)$.

Then $\frac{1}{\sum_{i=1}^r |s_i|^2}$ defines a possible sing metric $h_B$ on $K_X+L$.

i.e., $e \in (K_X + L)_x$

$$|e|^2_{h_B}(x) := \frac{|e|^2(x)}{\sum_{i=1}^{r} |s_i|^2(x)}$$

$$\Rightarrow \boxed{i\Theta_{h_B}(K_X + L) \geq 0 \text{ on } X.}$$

$\forall x \in X$

**Prop** (Extremal property), $e \in (K_X + L)_x$

$$\boxed{|e|^2_{h_B}(x)} = \boxed{\inf_{\substack{s \in H^0(X, K_X+L) \\ |s|_h = 1}} \frac{|e|^2(x)}{|s|^2(x)}}$$

~~Exo~~ Exo. Indicative $|s|_h = 1 \Leftrightarrow s = \sum_{i=1}^{r} a_i s_i$

$a_i \in \mathbb{C}$ $\sum_{i=1}^{r} |a_i|^2 = 1$

+ Cauchy $\Rightarrow$ Prop.

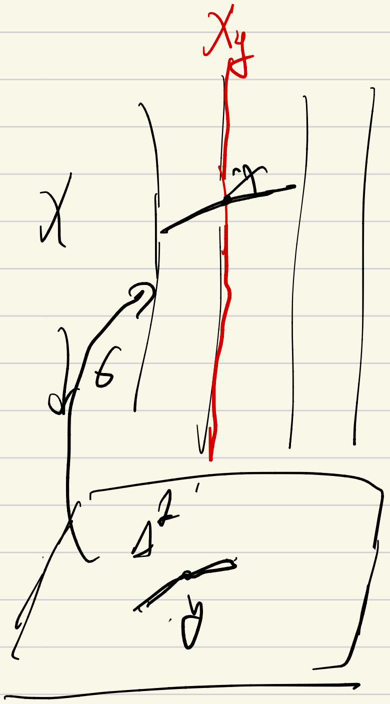
$\boxed{\text{Cor}}$ · $h_B$ is indepent of the choice of orthornal base.

· $\forall \boxed{s \in H^0(X, K_X + L)}$ $|s|^2_{h_B}(x) \leq C$ $\forall x \in X$.

**proof of Thm 2**: $\boxed{\pi: X \longrightarrow Y \text{ submersion.}}$ $L \to X$

$i\Theta_{h_L}(L) \geq 0$. $\pi_*(K_{X/Y} + L) \neq 0$.

We want to construct a metric $h_B$ on $K_{X/Y}+L$,



s.t. $i\Theta_{h_B}(K_{X/Y}+L)\geq 0$.

Let $e\in (K_{X/Y}+L)_x$

$$|e|^2_{h_B} := \frac{|e|^2(x)}{\sum_{i=1}^{r}|s_i|^2(x)}$$

where $\{s_1,\dots,s_r\}$ is an orthonormal basis of $H^0(X_y, K_{X/Y}+L), h)$

$y=\pi(x)$.

In other words, $h_B$ is a fiberwise Bergman-kernel metric.

We want to prove: $\boxed{i\Theta_{h_B}(K_{X/Y}+L)\geq 0}$ on $X$

Cor $\Rightarrow$ $\forall s\in H^0(X_y, K_{X/Y}+L)$

$|s|_{h_B}\leq C$

By construction, $i\Theta_{h_B}(K_{X/Y}+L)|_{X_y}\geq 0$.

Let $\boxed{\Delta^1 \xrightarrow{p} Y}$ be a unit disc in $Y$, and $p(0)=y$. Let $\sigma: \Delta^1\longrightarrow X$ s.t. $\sigma(0)=x$, $\pi\circ\sigma=p$

Let $e$ be a base of $K_{X/Y}+L$ near $x$.

$$|e|^2_{h_B} = e^{-\varphi_B} \qquad \varphi_B \text{ is function defined near } x.$$

$i\Theta_{h_B} \geq 0 \iff \varphi_B$ is psh.

near $x$

We need $\quad \varphi_B(x) \leq \frac{1}{4}\int_{\sigma(\Delta)} \varphi_B \, dz \wedge d\bar{z}$

Extremal property.

$\exists\ s \in H^0(X_y, K_{X/Y}+L)$, s.t. $|s|_h = 1$

$\varphi_B(x) = \ln\left|\frac{s}{e}\right|^2(x)$

By O-T: $\exists\ \tilde{s} \in H^0(\pi^{-1}(\Delta), K_{X/Y}+L)$

- $\int_\Delta |\tilde{s}|^2_h \, dz \wedge d\bar{z} \leq \pi$

- $\tilde{s}(0) = s$.

$\Rightarrow \ln\left|\frac{\tilde{s}}{e}\right|$ is psh function.

$$\varphi(x) = \ln\left|\frac{\tilde{s}}{e}\right|(x) \leq \frac{1}{\pi}\int_{\sigma(\Delta)} \ln\left|\frac{\tilde{s}}{e}\right| \sigma^*(dz \wedge d\bar{z})$$

$$\le \underbrace{\frac{1}{\pi}\int_{\Delta} \ln |s|_h^2 \, dz \wedge d\bar{z}} + \underbrace{\frac{2}{\pi}\int_{\sigma(\Delta)} \varphi_B \, \sigma^*(dz \wedge d\bar{z})},$$

$$\le \ln\Big(\underbrace{\frac{2}{\pi}\int_{\Delta} |s|_h^2 \, dz \wedge d\bar{z}}_{\le 1}\Big)$$

$$\le 0$$

$$\Rightarrow \varphi_B(x) \le \frac{1}{\pi}\int_{\sigma(\Delta)} \varphi_B \, \sigma^*(dz \wedge d\bar{z}) \qquad \square$$

$\Rightarrow \varphi_B$ is psh $\Rightarrow i\Theta_{h_B} \ge 0$.

<u>Next step</u>: generalise <u>Thm 1, 2</u> to <u>proper fibration case</u>.

<u>Thm</u> (Rieman extension thm for psh):

- $X$ complex mfd $Z \subset X$ subvariety.
- Let $\varphi$ be a psh on $X \setminus Z$.

(1). If $\operatorname{cod} Z = 1$, and $\boxed{\varphi \le C}$ on $X \setminus Z$.

$\Rightarrow$ we can extend $\varphi$ to $X$, and $\varphi$ is psh on $X$.

(2) If $\operatorname{cod}_X Z \geq 2$, $\Rightarrow$ $\varphi$ can be extende to $X$

$\varphi$ psh on $X$.

Demailly's book.

Rk. $f$ holo on $X \setminus Z$.

If $|f| \leq C$ $\Rightarrow$ $f$ extend to $X$

$f$ holo on $X$

$\varphi = \ln|f|$ psh on $X \setminus Z$

Cor: $L \to X$ holo line bundle. $Z \subset X$ subvariety.

$\exists\, h_L$ on $L|_{X \setminus Z}$ s.t. $i\Theta_{h_L}(L) \geq 0$. on $X \setminus Z$.

Let $z \in Z$ be a generic pt.

and $e_L$ be a holo base of $L$ near $z$,

$|e_L|^2_{h_L} = e^{-\varphi_L}$ $\varphi_L$ define on $X \setminus Z$

near $z$

If $\varphi_L \leq C$ $\Rightarrow$ $h_L$ can be extend to $X$,

$i\Theta_{h_L}(L) \geq 0$ on $X$.

$\Rightarrow L$ psef line bundle on $X$. $\square$