Critical exponent

Thursday, 18. August 2022 15:39

Kleinian gps with small critical exponent must be c.c.

joint w/ Beibei Liu

Set up: • $\mathbb{H}^n$ $(K\equiv -1)$

• $\mathrm{Isom}^+(\mathbb{H}^n) \cong SO^+(n,1)$

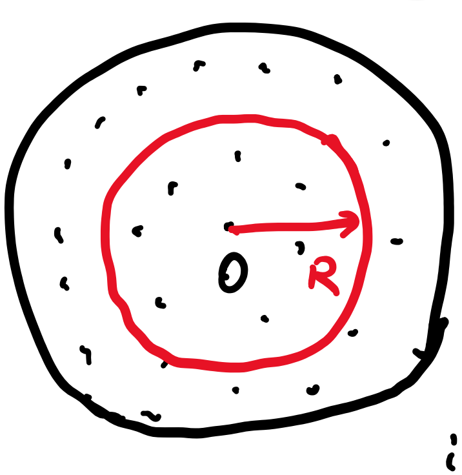
$n=2: \cong PSL_2(\mathbb{R})$

$n=3: \cong PSL_2(\mathbb{C})$ ...

• $\boxed{\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)}$ discrete subgp.

(Kleinian gps), f.g, torsion-free

$\Leftrightarrow$ $\mathbb{H}^n/\Gamma$ complete, Riemannian mfld, hyperbolic $(K\equiv -1)$



$\mathbb{H}^n$

$$\varlimsup_{R\to\infty} \frac{\log \#\{\gamma \mid d(\gamma o, o) < R\}}{R}$$

$:= \delta(\Gamma)$ critical exponent

(1) $\delta(\Gamma)$ independent on basept o.

(2) $0 \le \delta(\Gamma) \le n-1$

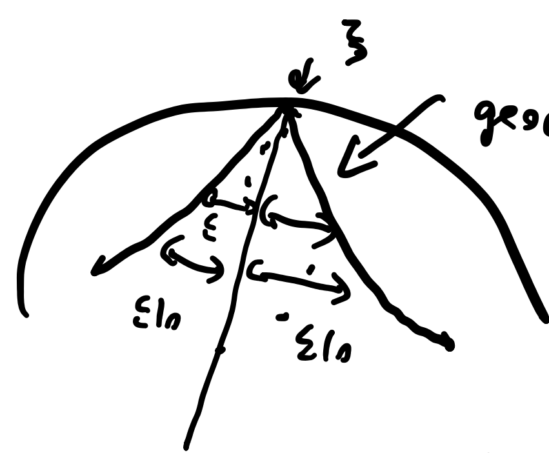
$\uparrow$ $\Gamma \cong \mathbb{Z} = \langle a\rangle$ hyperbolic

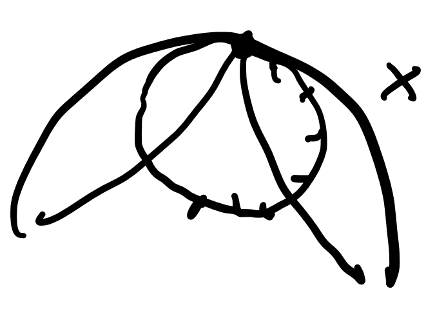
$\uparrow$ Γ lattice.

(3) $\delta(\Gamma) = \dim_H(\Lambda_c(\Gamma))$ ← conical limit set

Thm (Bishop-Jones) $\cap$

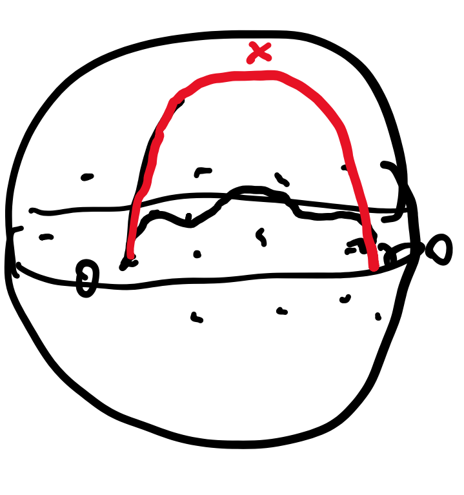
$\Lambda(\Gamma)$ ← limit set





(4) $$P_s(o) = \sum_{\gamma\in\Gamma} e^{-s\,d(o,\gamma o)} = \begin{cases} <\infty & s>\delta(\Gamma) \\ =\infty & s<\delta(\Gamma) \\ \infty/<\infty & s=\delta(\Gamma) \end{cases}$$

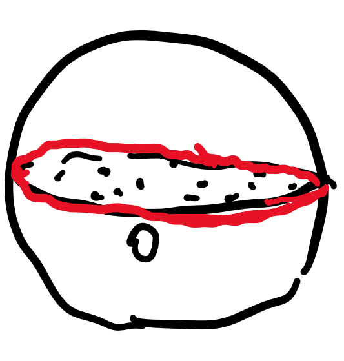
Def: $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ is convex cocpt (c.c.) if orbit map $\Phi: \Gamma \to \mathbb{H}^n$ is
(word metric)
$\gamma \mapsto \gamma o$
a quasi-isometric embedding.



$$\boxed{\Lambda_c(\Gamma) = \Lambda(\Gamma) \text{ iff } \Gamma \text{ is c.c.}}$$

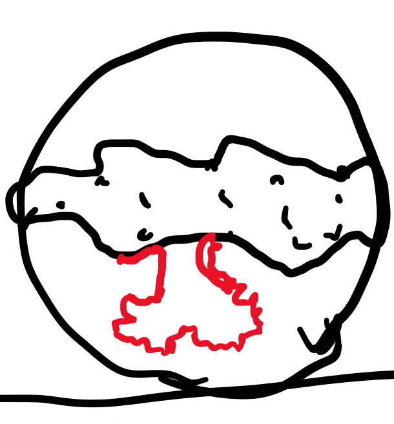
Example 1:
$$\begin{tikzcd} \Gamma = \pi_1(\Sigma_g) \arrow[r] \arrow[rr, bend right] & \mathrm{Isom}^+(\mathbb{H}^2) \arrow[r, hook] & \mathrm{Isom}^+(\mathbb{H}^n) \end{tikzcd}$$
$(g\geq 2)$

Fuchsian-representation



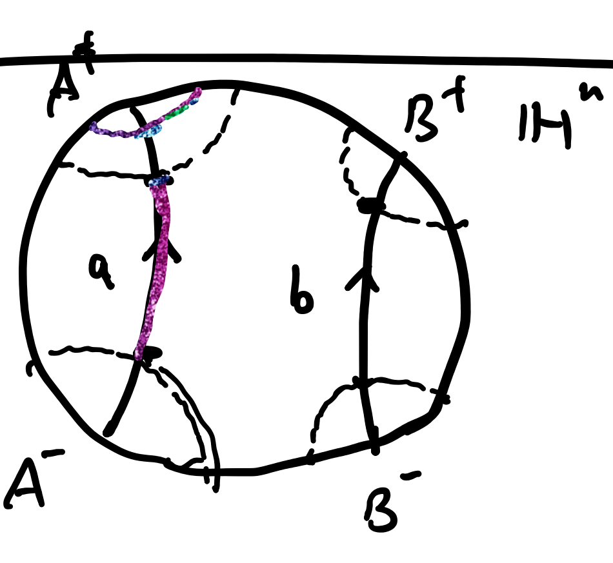
$$\underset{\overset{\|}{\delta(\Gamma)}}{\dim_H(\Lambda(\Gamma))} = \dim_{Top}(\Lambda(\Gamma)) = 1$$

Quasi-Fuchsian.



$\dim_T(\Lambda(\Gamma)) = 1$ $\quad$ $\underset{\overset{\|}{\delta(\Gamma)}}{\dim_H(\Lambda(\Gamma))} \geq 1$

Example 2:



$a(A^+, B^\pm) \subseteq A^+$ ‖ $b(B^+, A^\pm) \subseteq B^+$

$a^{-1}(A^-, B^\pm) \subseteq A^-$ ‖ $b^{-1}(B^-, A^\pm) \subseteq B^-$

Lemma (Ping-pong) $\langle a, b\rangle \cong F_2$

Pf: $$\underbrace{a^3 \underbrace{b \underbrace{a^{-1} \underbrace{b^2 (A^-)}_{\subseteq B^+}}_{\subseteq A^-}}_{\subseteq B^+}}_{A^+} \subseteq A^+$$

(Schottky gp) is c.c.

Possibly replace a, b by a^k, b^k $(k\to\infty)$

$\delta(\langle a^k, b^k\rangle) \to 0$

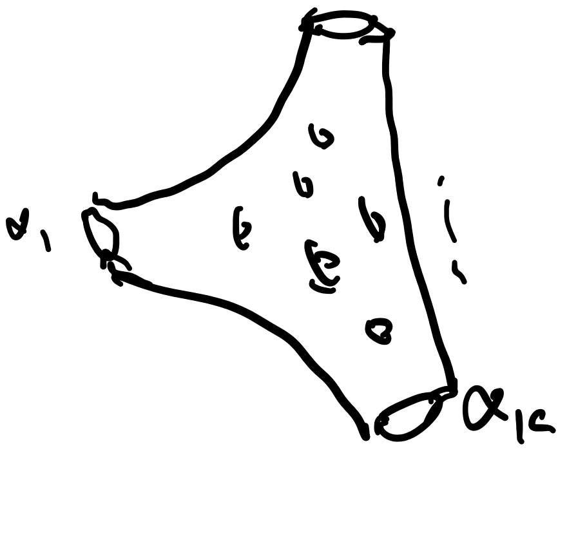
Conj (Kapovich, 09) Γ f.g. Kleinian gp, $\delta(\Gamma) < 1$, (with no parabolic elements).
then Γ is c.c.

Thm (Liu-W, 20') $\exists\, \varepsilon(n) > 0$,
Γ f.g. Kleinian gp, if $\delta(\Gamma) < \varepsilon(n)$,
then Γ is c.c.

Remark: if Γ is f.p, then Γ is free.
choose $\varepsilon(n) = 1/16$.

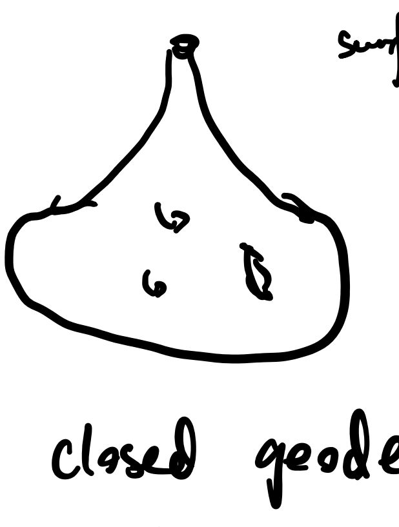
Pf: **Step1** (Differential geometric).

Thm: If $\delta(\Gamma) < 1$, then $\forall \{\alpha_1, \cdots, \alpha_k\}$ smooth loops in $M = \mathbb{H}^n/\Gamma$, $\sum a_i[\alpha_i] = 0$ in $H_1(M,\mathbb{Z})$



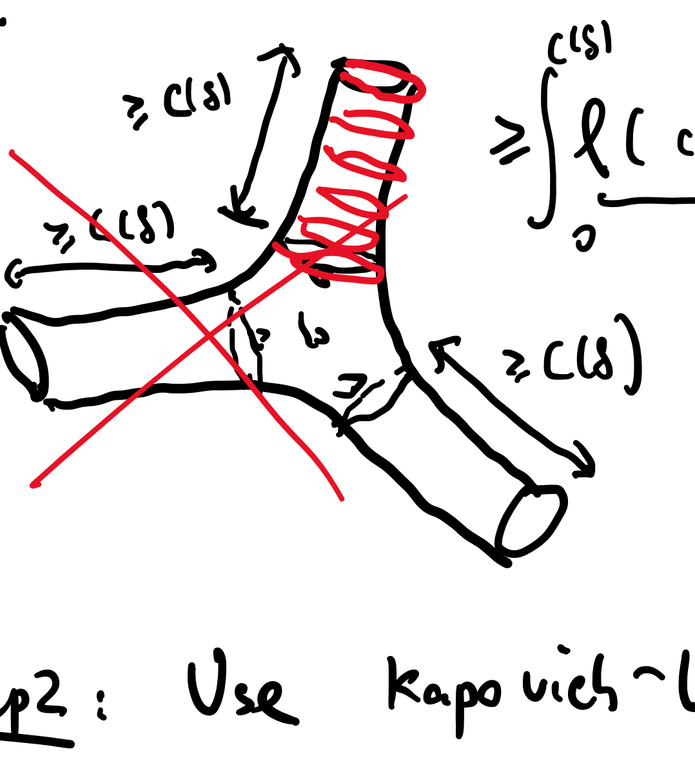
$\exists$ a smooth surface Σ.
s.t. $\partial\Sigma = \bigcup a_i\alpha_i$
and
(2)
$$\underline{A(\Sigma)} \leq \underset{\overset{\|}{\left(\frac{4}{1-\delta}\right)}}{C(\delta)} \cdot \underline{\ell(\partial\Sigma)}$$

"Counterexample":



surface $\mathbb{H}^2$
$\delta(\Gamma) = 1$.

Cor: if $\{\alpha_i\}$ closed geodesics.



$$\geq \int_0^{C(\delta)} \ell(\text{circle level } t)\, dt \geq C(\delta)\cdot\ell(\alpha_i)$$

Step2: Use Kapovich-Liu.

Thm (Kapovich-Liu): Γ is not c.c.
iff $\exists$ escaping closed geodesics on M.

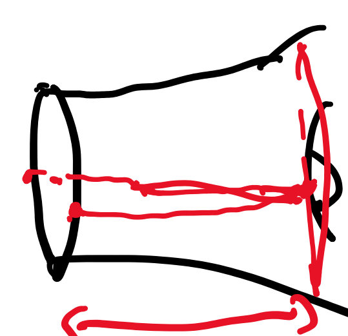
Prove by contradiction: if not c.c.
$\exists \{\alpha_i\}$ closed geod. $(\to\infty)$ by step 2

f.g. $\Rightarrow$ $H_1(M, \mathbb{Z})$ finit dim.

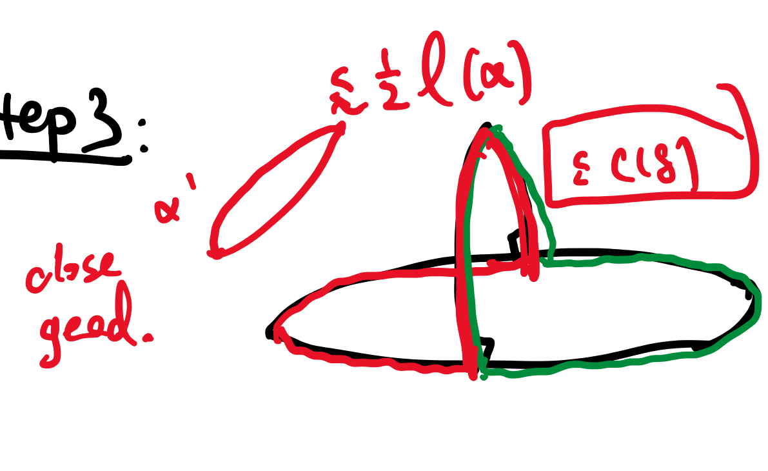
$\mathbb{R}$

k

+ step1 $\Rightarrow$ $\exists\{\alpha_i\}$ closed geod $(\to\infty)$
and $NJ(\alpha_i) \leq C(\delta)$

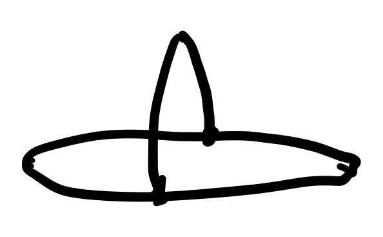


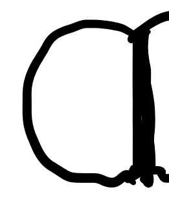
Step3:



cut along NJ, repeat step 1, 2, 3.

$\Rightarrow$ $\exists\{\alpha_i\}$ closed geod. $\to\infty$
$NJ \leq C(\delta)$
$\ell(\alpha_i) \lesssim C(\delta)$





$\Gamma_0 < \Gamma$
free

$$\underline{\delta} = \delta(\Gamma) \geq \delta(\Gamma_0) \geq \underline{C'(\delta)}$$

choose δ small enough. ↯. □
