Closed geodesics on most symmetric tori: monotonicity, positivity, and log-concavity of the shifted trace polynomials

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Aim To explore the ordering of the lengths (or traces) of all **closed geodesics** on the **most symmetric** hyperbolic **tori** with one cusp or one hole.

In the special case of the hyperbolic one-cusped torus (called the modular torus), this is related to the **Uniqueness Conjecture** for classical **Markoff numbers**.

C. Series, The geometry of Markoff numbers.
Math. Intelligencer 7(1985), no. 3, 20–29.

T. Cusick and M. Flahive, The Markoff and Lagrange Spectra.
American Mathematical Society, 1989.

Geometrically:

A hyperbolic one-holed torus is said to be **most symmetric** if there are **three** simple closed geodesics **of equal length** on the torus which intersect pairwise transversely at one point.

A **related** object:

A hyperbolic pair of pants is said to be **most symmetric** if the **three** boundary geodesics have the **same length**.

- **Corresponding results for the most symmetric pants.**

Hyperbolic plane H^2

Upper half-plane model of the hyperbolic plane is

$$\mathrm{H}^2 = \{(x, y) \in \mathbf{R}^2 \mid y > 0\},$$

with the Riemannian metric $ds^2 = y^{-2}(dx^2 + dy^2)$.

Group of orientation-preserving isometries of H^2:

$$\mathrm{Isom}^+(\mathrm{H}^2) \cong \mathrm{PSL}(2, \mathbf{R}) = \mathrm{SL}(2, \mathbf{R})/\{\pm I\},$$

acting as fractional linear transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d}.$$

Types of orientation-preserving isometries of H^2

Suppose $\pm A \in \mathrm{PSL}(2, \mathbf{R})$ where $A \neq I$.

As an orientation preserving isometry of H^2:

- A is of **hyperbolic type** $\Longleftrightarrow |\mathrm{tr}(A)| > 2$;

  in this case, $|\mathrm{tr}(A)| = 2\cosh(L_A/2)$ for $L_A > 0$.

- A is of **elliptic type** $\Longleftrightarrow |\mathrm{tr}(A)| < 2$;

  in this case, $\pm\mathrm{tr}(A) = 2\cos(\Theta_A/2)$ for $\Theta_A \in \mathbf{R} \bmod 2\pi$.

- A is of **parabolic type** $\Longleftrightarrow |\mathrm{tr}(A)| = 2$;

  in this case, we may say $L_A = 0$ or $\Theta_A = 0$.

Hyperbolic surfaces

S: A complete hyperbolic surface of finite type.

Associated **developing maps** and **holonomy representations**:

$$\mathrm{dev} : \tilde{S} \to \mathrm{H}^2,$$

$$\mathrm{hol} : \pi_1(S) \to \mathrm{Isom}(\mathrm{H}^2).$$

If S is orientable, hol maps to $\mathrm{Isom}^+(\mathrm{H}^2) \cong \mathrm{PSL}(2, \mathbf{R})$.

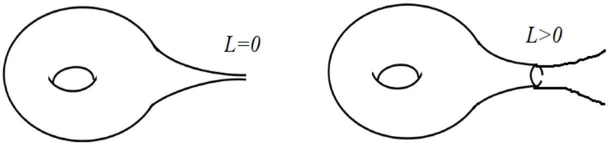
If possible, one works with a lift holonomy representation:

$$\rho : \pi_1(S) \to \mathrm{SL}(2, \mathbf{R}).$$

Hyperbolic torus with one geometric boundary component

Let $S_{1,1}$ be a complete hyperbolic torus with one **geometric boundary component**, which is

1) a cusp, or
2) a simple closed geodesic of length $L > 0$.

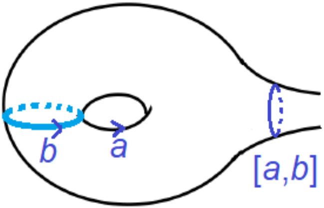


The fudamental group of $S_{1,1}$

Fudamental group of $S_{1,1}$ is a free group of rank 2:

$$\pi_1(S_{1,1}) = \langle a, b \mid - \rangle,$$

where a, b are represented by two simple closed curves on $S_{1,1}$ which intersect transversely at one point.



Lifted holonomy of a hyperbolic torus $S_{1,1}$

Since $\pi_1(S_{1,1}) \cong \langle a, b \mid - \rangle$ is a free group, it is possible to obtain a lifted holonomy representation:

$$\rho : \pi_1(T) \to \mathrm{SL}(2, \mathbf{R})$$

by arbitrarily choosing the $\pm$ signs:

$$\rho(a) = \pm A \in \mathrm{SL}(2, \mathbf{R}),$$

$$\rho(b) = \pm B \in \mathrm{SL}(2, \mathbf{R}).$$

There are four such lifts. There is exactly one lift with

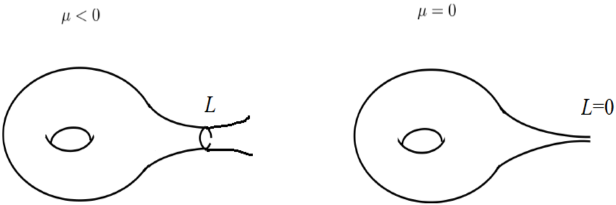
$$\mathrm{tr}\rho(a) > 2, \quad \mathrm{tr}\rho(b) > 2, \quad \mathrm{tr}\rho(ab) > 2.$$

Trace of the boundary loop

The **trace** τ **of the boundary loop** is given by

1) $\tau = -2\cosh(L/2) < -2$, or
2) $\tau = -2$.

Notation. $\mu := \tau + 2 \le 0$.



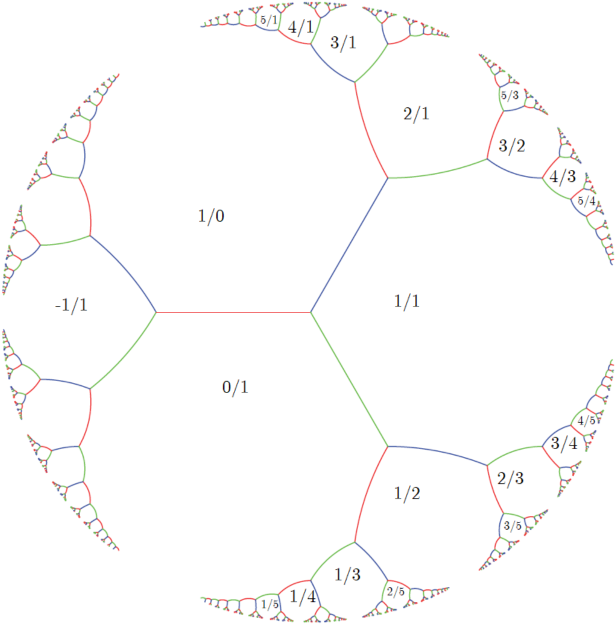
Simple closed curves and their slopes

The isotopy class of an unoriented essential **simple closed curve** on the 1-holed torus $S_{1,1}$ is determined by its **slope** $m/n \in \mathbf{Q}P^1$, where $m, n \in \mathbf{Z}$ are relatively prime.

Explicitly, if a and b are two simple closed curves on $S_{1,1}$ that intersect transversely at one point, and we set their slopes as $0/1$ and $1/0$, respectively, then the two simple closed curves ab and ab^{-1} have slopes $1/1$ and $-1/1$, respectively.

In this way, essential simple closed curves on $S_{1,1}$ are in one-to-one correspondence with their slopes $m/n \in \mathbf{Q}P^1$.

Examples: The simple closed curves aab, $aaab$, $aabab$ have slopes $1/2$, $1/3$, $2/3$, respectively.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2/5

Trace identities. For $A, B \in \mathrm{SL}(2, \mathbf{C})$,

$$\mathrm{tr}(A^{-1}) = \mathrm{tr}(A),$$

$$\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\,\mathrm{tr}(B).$$

Proof of the 2nd identity. Since $B \in \mathrm{SL}(2, \mathbf{C})$, we have

$$B + B^{-1} = I\,\mathrm{tr}(B),$$

$$AB + AB^{-1} = A\,\mathrm{tr}(B),$$

$$\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\,\mathrm{tr}(B).$$

Trace polynomials

- **Trace polynomial of a word**

Proposition. The trace of a word in $A, B \in \mathrm{SL}(2, \mathbf{C})$ is a polynomial in $\mathrm{tr}(A)$, $\mathrm{tr}(B)$ and $\mathrm{tr}(AB)$ with integer coefficients.

Proof. By induction on word length, and use trace identities. □

Trace polynomials

Notation. $x = \mathrm{tr}(A), \quad y = \mathrm{tr}(B), \quad z = \mathrm{tr}(AB).$

Examples: $\mathrm{tr}(I) = 2, \quad \mathrm{tr}(AA) = x^2 - 2, \quad \mathrm{tr}(AB^{-1}) = xy - z,$

$$\mathrm{tr}(ABA^{-1}B^{-1}) = x^2 + y^2 + z^2 - xyz - 2.$$

Lifted holonomy recovered from traces

Lifted holonomies can be recovered from the three traces

$$\mathrm{tr}(A), \ \mathrm{tr}(B), \text{ and } \mathrm{tr}(AB).$$

Goldman's formula: Given $x, y, z \in \mathbf{C}$, let

$$A = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \zeta^{-1} \\ -\zeta & y \end{pmatrix},$$

where $\zeta + \zeta^{-1} = z$. Then

$$\mathrm{tr}(A) = x, \quad \mathrm{tr}(B) = y, \quad \mathrm{tr}(AB) = z.$$

Trace of the simple closed curve of slope m/n

The **trace** of the simple closed curve **of slope** m/n is a polynomial

$$P_{m/n}(x, y, z).$$

Examples.

$$P_{0/1} = x, \quad P_{1/0} = y, \quad P_{1/1} = z.$$

$$P_{-1/1} = xy - z,$$

$$P_{1/2} = P_{0/1}P_{1/1} - P_{1/0} = xz - y,$$

$$P_{1/3} = P_{0/1}P_{1/2} - P_{1/1} = x^2z - xy - z,$$

$$P_{2/3} = P_{1/1}P_{1/2} - P_{0/1} = xz^2 - yz - x.$$

Parametrizing hyperbolic structures on $S_{1,1}$

The **marked hyperbolic structure** on $S_{1,1}$ is **parametrized by** the ordered triple $(x, y, z) \in (\mathbb{R}_{>2})^3$ which satisfies

$$x^2 + y^2 + z^2 - xyz = \mu,$$

where it is required that $\mu \leq 0$.

Recall: The **boundary loop** has **trace**

$$\tau = x^2 + y^2 + z^2 - xyz - 2,$$

and we write

$$\mu = \tau + 2 = x^2 + y^2 + z^2 - xyz.$$

Relative Teichmüller spaces

Relative Teichmüller space of $S_{1,1}$ consists of all marked complete hypebolic structures on $S_{1,1}$ with fixed boundary value:

$$\mathrm{Teich}_{1,1}(\mu), \quad \mu \leq 0.$$

Trace parametrization of $\mathrm{Teich}_{1,1}(\mu)$, $\mu \leq 0$:

$$\mathrm{Teich}_{1,1}^{\mathrm{trace}}(\mu) = \{(x, y, z) \in (\mathbb{R}_{>2})^3 \mid x^2 + y^2 + z^2 - xyz = \mu\}.$$

The most symmetric hyperbolic tori

- **What are the most symmetric hyperbolic tori?**

The **most symmetric hyperbolic tori** are those parametrized by

$$(t, t, t) \in \mathrm{Teich}_{1,1}^{\mathrm{trace}}(\mu), \quad t \geq 3.$$

Note that

$$\mu = t^2 + t^2 + t^2 - t^3 \leq 0.$$

- The triple $(3, 3, 3)$ gives the one-cusped **modular torus**.

Aim 1: Ordering of trace polynomials for all slopes

Aim 1. To explore the ordering of $P_{m/n}(t,t,t)$ for all slopes m/n.

By symmetry, it is enough to consider slopes $m/n \in [0/1,\ 1/1]$.

Our setting. $P_{0/1} = t, \quad P_{1/0} = t, \quad P_{-1/1} = t.$

Some simple examples.

$$P_{1/1} = t^2 - t,$$

$$P_{1/2} = P_{0/1}P_{1/1} - P_{1/0} = t^3 - t^2 - t,$$

$$P_{1/3} = P_{0/1}P_{1/2} - P_{1/1} = t^4 - t^3 - 2t^2 + t.$$

The polynomial $P_{m/n}(t,t,t)$ in t has both positive and negative coefficients, which makes the comparison not convenient.

- **Shifted trace polynomials**

Let

$$t = 3 + x, \quad x \geq 0.$$

We then obtain the **shifted trace polynomial**

$$p_{m,n}(x) := P_{m/n}(3+x, 3+x, 3+x).$$

Examples of shifted polynomials

Examples.

$$p_{1,1}(x) = x^2 + 5x + 6,$$

$$p_{1,2}(x) = x^3 + 8x^2 + 20x + 15,$$

$$p_{1,3} = x^4 + 11x^3 + 43x^2 + 70x + 39,$$

$$p_{2,3} = x^5 + 13x^4 + 66x^3 + 163x^2 + 194x + 87,$$

$$p_{2,5} = x^7 + 19x^6 + 151x^5 + 649x^4 + 1624x^3 + 2357x^2 + 1829x + 582.$$

The classical **Markoff numbers** are given by

$$M(m,n) = \frac{1}{3}\, p_{m,n}(0).$$

Classical Markoff numbers

Examples of Markoff numbers:

$$M(0,1)=1, \qquad M(1,2)=5, \qquad M(1,1)=2$$

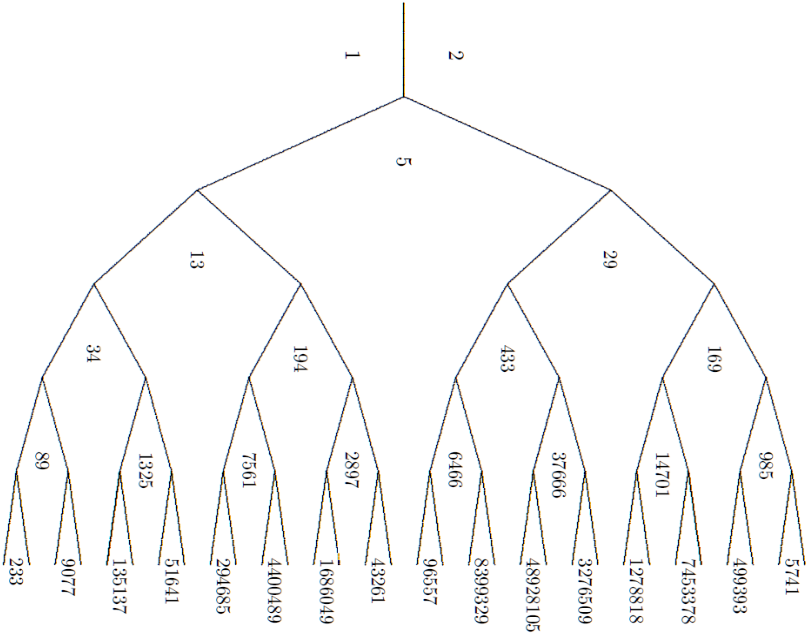
$$M(1,3)=13, \qquad M(2,3)=29$$

$$M(1,4)=34, \qquad M(2,5)=194, \qquad M(3,5)=433, \qquad M(3,4)=169$$

Algorithm for calculating Markoff numbers:

$$M(3,5)=3\,M(1,2)\,M(2,3)-M(1,1)$$

$$433=3\times 5\times 29-2$$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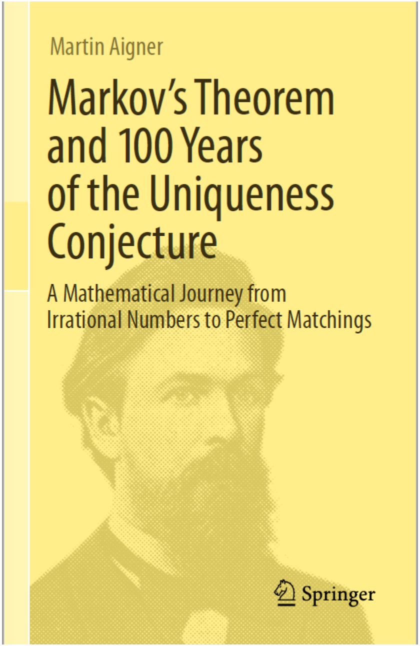

Martin Aigner
Markov's Theorem and 100 Years of the Uniqueness Conjecture
A Mathematical Journey from Irrational Numbers to Perfect Matchings
Springer

Aigner made **three conjectures** in his book "Markov's Theorem and 100 Years of the Uniqueness Conjecture" (Springer, 2013).

Aigner's Fixed Numerator Conjecture

Conjecture A1. $M(r,s) < M(r,s+1) < M(r,s+2) < \cdots$

$$\begin{array}{ccccccccccc} m_{1/2} & < & m_{1/3} & < & m_{1/4} & < & m_{1/5} & < & m_{1/6} & < & \dots \\ 5 & < & 13 & < & 34 & < & 89 & < & 233 & < & \dots \end{array}$$

$$\begin{array}{ccccccccccc} m_{2/3} & < & m_{2/5} & < & m_{2/7} & < & m_{2/9} & < & m_{2/11} & < & \dots \\ 29 & < & 194 & < & 1,325 & < & 9,077 & < & 62,210 & < & \dots \end{array}$$

Aigner's Fixed Denominator and Fixed Sum Conjectures

Aigner's Fixed Denominator Conjecture

Conjecture A2. $M(r,s) < M(r+1,s) < M(r+2,s) < \cdots$

Aigner's Fixed Sum Conjecture

Conjecture A3. $M(1,k-1) > M(2,k-2) > M(3,k-3) > \cdots$

Recent Progresses

Conjectures A1-A3 have been claimed to be **proved** recently.

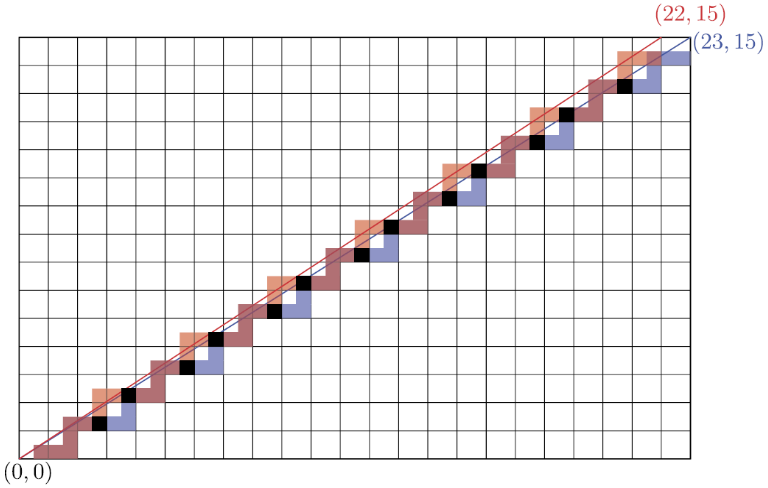
[0] M. Rabideaua and R. Schifflerb, Continued fractions and orderings on the Markov numbers, Adv. Math. 370 (2020), 107231. (Conjecture A1)

[1] K. Lee, L. Li, M. Rabideau, and R. Schiffler, On the ordering of the Markov numbers, arXiv2010.13010 (October 25, 2020).

[2] C. Lagisquet, E. Pelantova, S. Tavenas, L. Vuillon, On the Markov numbers: fixed numerator, denominator, and sum conjectures, arXiv2010.10335 (October 20, 2020).

[3] G. McShane, Convexity and Aigner's conjectures, arXiv:2101.03316 (January 9, 2021).

M. Rabideau, R. Schiffler / Advances in Mathematics 370 (2020) 107231



Words for (multiple round) simple closed curves

- Balanced words = Words for simple closed curves

Shifted trace polynomials $p_w(x)$

Our setting. Let $A, B \in \mathrm{SL}(2, \mathbf{R})$ be such that (where $x \geq 0$)

$$\mathrm{tr}(A) = \mathrm{tr}(B) = \mathrm{tr}(AB^{-1}) = 3 + x.$$

Then for any **word** $w = w(A, B)$, we define a polynomial p_w by

$$p_w(x) = \mathrm{tr}(w(A, B)).$$

Words of repeated simple closed curves. For integers $m \geq 0$ and $n \geq 0$ (need not be relatively prime), we have a positive word

$$w_{m,n} = w_{m,n}(A, B)$$

which represents a repeated simple closed curve on $S_{1,1}$.

Shifted trace polynomials $p_{m,n}(x)$

We then define **trace polynomial** $p_{m,n}$ by

$$p_{m,n}(x) = \mathrm{tr}(w_{m,n}(A, B))$$

for integers $m \geq 0$ and $n \geq 0$ (need not be relatively prime).

Examples.

$$p_{0,0}(x) = \mathrm{tr}(I) = 2,$$

$$p_{3,0}(x) = \mathrm{tr}(AAA) = x^3 + 9x^2 + 24x + 18,$$

$$p_{2,4}(x) = \mathrm{tr}(AABAAB) =$$

$$x^6 + 16x^5 + 104x^4 + 350x^3 + 640x^2 + 600x + 223.$$

Properties of coefficients of $p_w(x)$

Definition (**positive polynomial**) We say that a real polynomial is **positive** if all of its coefficients are positive.

Theorem (Li-Z.) For **every word** w in A and B, one of $\pm p_w$ is a **positive polynomial** in x.

Properties of coefficients of $p_w(x)$

Definition (**log-concavity**) A sequence of positive real numbers $\{a_j\}$ is said to be **log-concave** if, for all j,

$$a_j^2 \geq a_{j-1}a_{j+1}.$$

We have observed, for every word w in A and B, **log-concavity** of the sequence of coefficients $\{c_j\}$ of $p_w(x)$.

Log-concavity Conjecture:

For **every word** w in A and B, the sequence of coefficients of the shifted trace polynomial $p_w(x)$ is **log-concave**.

Strong succeeding order between polynomials

Definition (**Strong succeeding order** between real polynomials)

For real polynomials f and g, we say that

f **strongly succeeds** g

and write

$$f \succ g,$$

if their difference $f - g$ is a **positive** polynomial.

As a simple example,

$$x^3 + 2x^2 + 3x + 4 \succ x^3 + x^2 + x + 1.$$

Our conjectures Z1-Z4

Inspired by Aigner's conjectures, we made in 2020 the following conjectures for the polynomials $p_{m,n}$ with $0 \leq m \leq n$.

Conjecture Z1. $p_{m,n+1} \succ (2+x)\, p_{m,n}$.

Conjecture Z2. $p_{m,n} \succ (2+x)\, p_{m-1,n}$.

Conjecture Z3. $p_{m-1,n+1} \succ p_{m,n}$.

Conjecture Z4. $p_{m-2,n+2} + p_{m,n} \succ 2p_{m-1,n+1}$.

Remark. In the case $x = 0$, Conjectures Z1-Z3 imply A1-A3.

Aim 2: Ordering of shifted polynomials for all words

Aim 2. To understand the ordering of $p_w(x)$ for all words

$$w = w(A, B).$$

• For a pair of integers $m, n \geq 0$, we have positive **word** $w_{m,n}$ for repeated simple closed curves, and the shifted trace polynomial

$$p_{m,n} = p_{w_{m,n}}.$$

Positive words in between

- **Succeeding order for positive words in between**

Theorem (Li-Z.) If $w = w(A, B)$ is a positive word which consists of n A's and m B's such that w does not represent any repeated simple closed curve and w is not conjugate to A^nB^m, then

$$p_{A^nB^m} \succ p_w \succ p_{m,n}.$$

Examples.

$$p_{AABB} \succ p_{ABAB};$$

$$p_{AAABB} \succ p_{AABAB};$$

$$p_{AAABBB} \succ p_{AABBAB} \succ p_{ABABAB}.$$

$$p_{AAAABBB} \succ p_{AABAABB} \succ p_{AABABAB}.$$

More results

Theorem (Li-Z.) Conj. **Z1** is **true**: $p_{m,n+1} \succ (2+x)\,p_{m,n}$.

Theorem (Li-Z.) Conj. **Z2** is **true**: $p_{m,n} \succ (2+x)\,p_{m-1,n}$.

Theorem (Li-Z.) Conj. **Z3** is **true**: $p_{m-1,n+1} \succ p_{m,n}$.

- Conjecture Z4 is open. — For $x = 0$:

Min Huang, On the monotonicity of the generalized Markov numbers, arXiv:2204.11443v2

More results

We have confirmed Log-concavity Conjecture for positive words:

Theorem (Li-Z.) If w is a **positive word** in A and B, then the sequence of coefficients of the polynomial $p_w(x)$ is **log-concave**.

- Work in progress to prove the Log-concavity Conj. for all words.

Our choice of the matrices A and B

Let $x \geq 0$, and the matrices A and B be given by

$$A = \begin{pmatrix} \frac{x+3}{2} + \frac{\sqrt{x+1}}{2} & \frac{\sqrt{x+1}\sqrt{x+4}}{2} \\ \frac{\sqrt{x+1}\sqrt{x+4}}{2} & \frac{x+3}{2} - \frac{\sqrt{x+1}}{2} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{x+3}{2} - \frac{\sqrt{x+1}}{2} & \frac{\sqrt{x+1}\sqrt{x+4}}{2} \\ \frac{\sqrt{x+1}\sqrt{x+4}}{2} & \frac{x+3}{2} + \frac{\sqrt{x+1}}{2} \end{pmatrix}.$$

In the case where $x = 0$,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Matrix for a general word $w = w(A, B)$

The matrix for a general word $w = w(A, B)$ is of the form

$$\begin{pmatrix} \frac{p}{2} + \frac{q\sqrt{x+1}}{2} & \frac{r\sqrt{x+1}\sqrt{x+4}}{2} + \frac{s\sqrt{x+4}}{2} \\ \frac{r\sqrt{x+1}\sqrt{x+4}}{2} - \frac{s\sqrt{x+4}}{2} & \frac{p}{2} - \frac{q\sqrt{x+1}}{2} \end{pmatrix},$$

where $p = p(x), q = q(x), r = r(x), s = s(x)$ are polynomials.

Thanks for attention

THANKS FOR ATTENTION